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ESTIMATES FOR SMOOTH ABSOLUTELY MINIMIZING LIPSCHITZ EXTENSIONS

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ABSTRACT. I present some elementary maximum principle arguments, establishing interior gradient bounds and Harnack inequalities for both u and $|Du|$, where u is a smooth solution of the degenerate elliptic PDE $\Delta_\infty u = 0$. These calculations in particular extend to higher dimensions G. Aronsson's assertion [2] that a nonconstant, smooth solution can have no interior critical point.

1. INTRODUCTION

G. Aronsson initiated in [1], [2] investigation of highly degenerate elliptic boundary value problem:

$$u_{x_i} u_{x_j} u_{x_i x_j} = 0 \text{ in } U \tag{1}$$

$$u = g \text{ on } \partial U, \tag{2}$$

where U is a bounded, connected, open subset of $\mathbb{R}^n$, $g : \partial U \to \mathbb{R}$ is a given Lipschitz function, and $u : \bar{U} \to \mathbb{R}$ is the unknown.

The PDE (1) arises naturally if we consider "optimal" Lipschitz extensions of g into U.

A function g defined on ∂U has in general many extensions into $\bar{U}$ which preserve its Lipschitz constant. Aronsson proposed trying to find a "best" Lipschitz extension u, characterized by the property that for each subdomain $V \subset U$, the Lipschitz constant of u within V equals the Lipschitz constant of u restricted to ∂V. More precisely, and following Jensen [6], let us say $u \in W^{1,\infty}(U)$ is an *absolutely minimizing Lipschitz extension* of g into U provided (2) holds and also

$$\|Du\|_{L^\infty(V)} \leq \|D\tilde{u}\|_{L^\infty(V)} \tag{3}$$

for each open set $V \subset U$ and each $\tilde{u} \in W^{1,\infty}(V)$ such that

$$u - \tilde{u} \in W_0^{1,\infty}(V). \tag{4}$$

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See Jensen [6] for more discussion, in particular concerning the equality

$$\|Du\|_{L^\infty(V)} = \sup_{\substack{x,y\in\bar{V} \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{d_V(x,y)} \right\},$$

$d_V(x,y)$ denoting the distance from x to y within V.

As noted by Aronsson, any smooth absolutely minimizing Lipschitz extension solves the PDE (1) within U, and Jensen provides a somewhat different proof. The best insight for this equation is had by considering instead of (1), (2) the corresponding boundary-value problem for the p-Laplacian:

$$\operatorname{div}(|Du_p|^{p-2}Du_p) = 0 \text{ in } U, \tag{5}$$

$$u_p = g \text{ on } \partial U, \tag{6}$$

when $n < p < \infty$. This is the Euler–Lagrange equation for the variational problem of minimizing the energy $\|D\tilde{u}\|_{L^p(U)}$ among all $\tilde{u} \in W^{1,p}(U)$ with $\tilde{u} = g$ on ∂U. In particular

$$\|Du_p\|_{L^p(V)} \leq \|D\tilde{u}\|_{L^p(V)} \tag{7}$$

for each open $V \subset U$ and each $\tilde{u}$ such that $u - \tilde{u} \in W_0^{1,p}(V)$. Assuming u_p is smooth and $|Du_p| \neq 0$, we may rewrite (5) to read

$$\frac{1}{(p-2)}\Delta u_p + \frac{u_{p,x_i}u_{p,x_j}}{|Du_p|^2}u_{p,x_ix_j} = 0. \tag{8}$$

Suppose also we knew that as $p \to \infty$, the function u_p converge in some sufficiently strong sense to a limit u. Formally passing to limits in (7), we would expect u to be an absolutely miminizing Lipschitz extension, and passing to limits in (8) we expect as well u to solve the PDE (1).

R. Jensen in his important paper [6] has made these insights rigorous. In addition, he has proved that (a) any absolutely minimizing Lipschitz extension is a weak solution of (1), and (b) any weak solution is unique. (Here "weak solution" means a solution in the so-called viscosity sense, cf. Crandall–Ishii–Lions [4]).

In view of the construction of absolutely minimizing Lipschitz extensions as limits of solution of the p-Laplacian, it seems reasonable to define, at least at points where $|Du| \neq 0$,

the nonlinear operator

$$\Delta_\infty u = \frac{u_{x_i}u_{x_j}}{|Du|^2}u_{x_ix_j} \tag{9}$$

as the "∞-Laplacian".

This paper is a small contribution to the further study of smooth solutions of the highly degenerate elliptic PDE $\Delta_\infty u = 0$, and, equivalently, of smooth absolutely minimizing Lipschitz extensions. I provide some elementary maximum principle arguments establishing interior sup-norm bounds on both $|D(\log u)|$ (if $u > 0$) and $|D(\log|Du|)|$. These imply in particular Harnack inequalities for u and $|Du|$. One consequence is that if u is not constant, then $|Du|$ can never vanish. This is an extension to dimensions $n \geq 3$ of a corresponding assertion of Aronsson in $n = 2$; cf. also Fuglede [5].

I should point out explicitly however that in general the $\Delta_\infty u = 0$ does not admit smooth solutions, and consequently the calculations presented here, although I think interesting, have limited applicability in practice. For instance, Aronsson in

[3] has constructed a C^1 nonconstant weak solution, which does indeed possess an interior critical point. This example shows that it is not merely a question of finding some reasonable approximation structure to which to modify the computation from §3.

My calculations estimating $|D(\log |Du|)|$ are somewhat reminiscent of standard computations for minimal surfaces. This suggests comparison of the PDE $\Delta_\infty u = 0$ with the "dual" equation $\Delta_1 u = 0$, where

$$\Delta_1 u = \left(\delta_{ij} - \frac{u_{x_i} u_{x_j}}{|Du|^2}\right) u_{x_i x_j} = |Du| \text{ div}\left(\frac{Du}{|Du|}\right).$$

Note that the operator Δ_1 is degenerate, but only in the one direction normal to each level set. By contrast, the operator Δ_∞ is nondegenerate only in this direction. Observe also

$\Delta_1 u = 0$ is a "geometric" equation, since it says that the level sets of u have zero mean curvature (at least in regions where u is smooth and $|Du| > 0$). This fact suggests that the PDE $\Delta_\infty u = 0$ is somehow strongly "nongeometric", or rather that all its geometric information concerns not the level sets of u, but rather the curves normal to level sets. The concluding remark in §3 makes this comment a bit more precise.

2. Interior gradient bounds, Harnack inequality for u

In this section we present a very simple proof of interior gradient bounds and a Harnack inequality for u.

Theorem 2.1. *Let u be a C^2 solution of*

$$\Delta_\infty u = 0 \text{ in } U. \tag{10}$$

(i) *There exists a constant C such that*

$$|Du(x_0)| \leq C\|u\|_{L^\infty(U)} \, dist\, (x_0, \partial U)^{-1} \tag{11}$$

for each point $x_0 \in U$.

(ii) *Suppose also $u \geq 0$. Then for each connected open set $V \subset\subset U$, there exists a constant $C = C(V)$ such that*

$$\sup_V u \leq C \inf_V u\,. \tag{12}$$

(iii) *In particular, if U is connected and $u > 0$ at some point in U, then $u > 0$ everywhere in U. In this case, we have the estimate*

$$\sup_V \left(\frac{|Du|}{u}\right) \leq C\,,$$

C depending only on n and $dist(V, \partial U)$.

Proof. 1. Write $v = |Du|$ and suppose for the moment we know $v \neq 0$ in U. Then (10) implies

$$u_{x_i} v_{x_i} = 0 \text{ in } U. \tag{13}$$

Define

$$w = \zeta \Phi(u, v),$$

where $\zeta \in C_c^\infty(U)$ and $\Phi \in C^2(\mathbb{R}^2)$ are smooth nonnegative functions, to be selected below. If w attains its maximum over $\bar{U}$ at a point $x_1 \in U$, we have

$$\zeta \Phi_u u_{x_i} + \zeta \Phi_v v_{x_i} = -\zeta_{x_i} \Phi \quad (1 \le i \le n)$$

at x_1. Multiply by

$$\nu^i = u_{x_i}/|Du| = u_{x_i}/v \quad (1 \le i \le n)$$

and sum on i, recalling (13) to deduce

$$\zeta \Phi_u v = -(D\zeta \cdot \nu)\Phi$$

at x_1. Consequently,

$$\zeta |\Phi_u| v \le |D\zeta| \Phi \tag{14}$$

at x_1.

2. Now take

$$\Phi(u,v) = e^{lu} v \quad (l > 0).$$

Then (14) reads

$$l\zeta(x_1) v(x_1) \le |D\zeta(x_1)|. \tag{15}$$

Fix $x_0 \in U$ and take $\zeta \in C_c^\infty(U)$ such that $\zeta(x_0) = 1$, $|D\zeta| \le 2 \operatorname{dist}(x_0, \partial U)^{-1}$.

As $w = \zeta\Phi$ attains its maximum at x_1, we see that

$$\begin{aligned} |Du(x_0)| &= v(x_0) \\ &\le \frac{2}{\lambda} e^{2\lambda \|u\|_{L^\infty}} \operatorname{dist}(x_0, \partial U)^{-1}. \end{aligned}$$

Set

$$l = \|u\|_{L^\infty}^{-1}$$

to prove estimate (11).

3. Next, assume $u \ge 0$, fix $\delta > 0$, and take

$$\Phi(u,v) = \frac{v}{u+\delta}.$$

Then (14) implies

$$\zeta \frac{v^2}{(u+\delta)^2} \le |D\zeta| \frac{v}{u+\delta}$$

at x_1. Consequently

$$w(x_1) = \frac{\zeta(x_1) v(x_1)}{u(x_1)+\delta} \le |D\zeta(x_1)|. \tag{16}$$

Given any ball $B \subset\subset U$, select ζ so that $\zeta \equiv 1$ on B, $|D\zeta| \le C \operatorname{dist}(B, \partial U)^{-1}$.

As $w = \zeta\Phi$ attains its maximum at x_1, we conclude from (16) that

$$\sup_B \left(\frac{v}{u+\delta} \right) \le C,$$

in constant C depending only on $\operatorname{dist}(B, \partial U)$. But $v = |Du|$, and so

$$\|D(\log(u+\delta))\|_{L^\infty(B)} \le C. \tag{17}$$

Now take any pair of points $x_1, x_2 \in B$. Let P denote the path

$$\{t x_2 + (1-t)x_1 \mid 0 \le t \le 1\}.$$

Then

$$\begin{aligned}\log(u(x_2)+\delta)-\log(u(x_1)+\delta) &= \int_0^1 \frac{d}{dt}[\log(u(tx_2+(1-t)x_1)+\delta)]\,dt\\ &= \int_0^1 D(\log(u(\cdot)+\delta))\cdot(x_2-x_1)\,dt\\ &\leq C\ \mathrm{diam}(B),\end{aligned}$$

according to (17). Consequently

$$u(x_2)+\delta \leq (u(x_1)+\delta)e^{C\ \mathrm{diam}\ (B)}.$$

Letting $\delta \to 0$, we deduce

$$u(x_2) \leq Cu(x_1)$$

for some constant C and each $x_1, x_2 \in B$. If $V \subset\subset U$ is connected, we cover V with balls and iteratively apply the foregoing result to each B, finally to deduce

$$u(x_2) \leq Cu(x_1)$$

for each pair of points $x_1, x_2 \in V$, the constant C depending only on V.

4. Finally, we remove the restriction $v = |Du| > 0$. For this define

$$\tilde{u}(\tilde{x}) = u(x) + \varepsilon x_{n+1},$$

where $\tilde{x} = (x_1, \dots, x_{n+1}) = (x, x_{n+1})$, $\varepsilon > 0$. Then $\tilde{u}$ is a C^2 solution of

$$\tilde{u}_{x_i}\tilde{u}_{x_j}\tilde{u}_{x_ix_j} = 0 \text{ in } \tilde{U},$$

$\tilde{U} = U \times \mathbb{R}$, and $|D\tilde{u}| \geq \varepsilon > 0$. Apply the calculations in steps 1-3 to $\tilde{u}$ in place of u. □

Remark. It is somewhat surprising in light of the extremely strong degeneracy of the nonlinear operator Δ_∞ that (smooth) solutions verify the interior gradient bound (11) and the Harnack inequality (12). Such estimates are usually the hallmarks of averaging effects resulting from uniform ellipticity. It is therefore perhaps worth noting that solutions do not in general satisfy the strong maximum principle. For example, let $u(x) = |x|$ and $\tilde{u}(x) = x_n$, and take U to be the open ball of radius one, centered at the point $(0, \dots, 2)$. Then $u, \tilde{u}$ are C^∞ solutions in U, $u \geq \tilde{u}$ in ∂U, but $u = \tilde{u}$ in U along the line $x' = 0$, $x' = (x_1, \dots, x_{n-1})$. This example shows also the Harnack inequality (12) is false if we "tilt" coordinates: it is *not* true that

$$\sup_V (u - L) \leq C \inf_V (u - L)$$

for each linear function L. □

3. Harnack inequality for $|Du|$

Our goal next is to establish a Harnack inequality for $v = |Du|$, and in particular to show a smooth, nonconstant u cannot have any critical point.

Theorem 3.1. *Let u be a C^4 solution of*

$$\Delta_\infty u = 0 \text{ in } U. \tag{18}$$

(i) Then for each smooth, connected $V \subset\subset U$ there exists a constant $C = C(V)$ such that

$$\sup_V |Du| \leq C \inf_V |Du|. \tag{19}$$

(ii) In particular, if U is connected and $|Du| > 0$ at some point in U, then $|Du| > 0$ everywhere in U. In this case

$$\sup_V \left(\frac{|D|Du||}{|Du|} \right) \le C,$$

C depending only on n and dist$(V, \partial U)$.

Proof. 1. Assume first $v = |Du| > 0$ everywhere in U. As above we write

$$\nu^i = u_{x_i}/|Du| = u_{x_i}/v \quad (1 \le i \le n), \tag{20}$$

and also write

$$h_{ij} = \nu^i \nu^j, \; g_{ij} = \delta_{ij} - \nu^i \nu^j \quad (1 \le i, j \le n). \tag{21}$$

Notice

$$\nu^i_{x_j} = \frac{1}{v} g_{ik} u_{x_k x_j} \quad (1 \le i, j \le n), \tag{22}$$

$$v_{x_i} = \nu^j u_{x_j x_i} \quad (1 \le i \le n). \tag{23}$$

Observe further that the PDE (18) says

$$\nu^i v_{x_i} = 0. \tag{24}$$

2. We derive a PDE v satisfies. We first differentiate (24) with respect to x_j and then utilize (22) to find

$$\nu^i v_{x_i x_j} = -\frac{1}{v} g_{ik} u_{x_k x_j} v_{x_i}.$$

Consequently

$$\begin{aligned} h_{ij} v_{x_i x_j} &= -\frac{\nu^j}{v} g_{ik} u_{x_k x_j} v_{x_i} \\ &= -\frac{g_{ik}}{v} v_{x_k} v_{x_j} \text{ by (23)} \\ &= -\frac{|Du|^2}{v} \text{ by (24)}. \end{aligned}$$

Therefore

$$-h_{ij} v_{x_i x_j} = |A|^2 v, \tag{25}$$

where we have written

$$|A|^2 = \frac{|Dv|^2}{v^2}. \tag{26}$$

3. Next we compute a differential inequality satisfied by $z = |A|^2$. For this, first write $w = \log v$. Then equation (25) becomes, in light of (24),

$$-h_{ij} w_{x_i x_j} = |A|^2 = z. \tag{27}$$

Now $z = |Dw|^2$, and so

$$z_{x_i x_j} = 2 w_{x_k x_i} w_{x_k x_j} + 2 w_{x_k} w_{x_k x_i x_j} \quad (1 \le i, j \le n).$$

Hence

$$-h_{ij} z_{x_i x_j} = -2\nu^i \nu^j w_{x_k x_i} w_{x_k x_j} + 2 w_{x_k} (-h_{ij} w_{x_k x_i x_j}). \tag{28}$$

We differentiate (27) with respect to x_k and substitute above, thereby deducing

$$-h_{ij} z_{x_i x_j} = -2\nu^i \nu^j w_{x_k x_i} w_{x_k x_j} + 2 w_{x_k} z_{x_k} 4 w_{x_k} \nu^j \nu^i_{x_k} w_{x_i x_j}\,. \tag{29}$$

Now (24) implies

$$\nu^j w_{x_j} = 0, \tag{30}$$

and so

$$\nu^j w_{x_i x_j} = -\nu^j_{x_i} w_{x_j} \quad (1 \le i \le n).$$

We can therefore rewrite the last term in (29) as

$$4 w_{x_k} \nu^j \nu^i_{x_k} w_{x_i x_j} = -4 w_{x_k} w_{x_j} \nu^j_{x_i} \nu^i_{x_k}. \tag{31}$$

Next, we return to (22) and compute

$$\begin{aligned} \nu^j_{x_i} \nu^i_{x_k} &= \frac{1}{v^2} g_{jm} u_{x_m x_i} g_{il} u_{x_l x_k} \\ &= \frac{1}{v} g_{jm} \nu^l_{x_m} u_{x_l x_k} \\ &= \frac{1}{v} g_{jm} \nu^l_{x_m} g_{lj} u_{x_j x_k} \\ &= g_{jm} \nu^l_{x_m} \nu^l_{x_k}, \end{aligned}$$

the penultimate equality holding since $\nu^l \nu^l_{x_m} = 0$. Inserting this computation into (31) yields

$$\begin{aligned} 4 w_{x_k} \nu^j \nu^i_{x_k} w_{x_i x_j} &= -4 w_{x_k} w_{x_j} g_{jm} \nu^l_{x_m} \nu^l_{x_k} \\ &= -4 w_{x_k} w_{x_m} \nu^l_{x_m} \nu^l_{x_k} \text{ by (30)} \\ &\le 0. \end{aligned}$$

Consequently, (29) implies

$$-h_{ij} z_{x_i x_j} \le -2 \nu^i \nu^j w_{x_k x_i} w_{x_k x_j} + 2 w_{x_k} z_{x_k}. \tag{32}$$

But (27) tells us

$$\begin{aligned} z^2 &= (\nu^i \nu^j w_{x_i x_j})^2 \\ &\le \nu^i w_{x_k x_i} \nu^j w_{x_k x_j}. \end{aligned}$$

Thus from (32) we discover the differential inequality

$$-h_{ij} z_{x_i x_j} \le -2z^2 + 2 w_{x_k} z_{x_k}. \tag{33}$$

4. We intend next to deduce from (33) an interior estimate on z. This is possible owing to the z^2 term in (33). Indeed, let $\zeta \in C^\infty_c(U)$, $0 \le \zeta \le 1$, and write $r = \zeta^4 z$. Then

$$r_{x_i} = \zeta^4 z_{x_i} + 4\zeta^3 \zeta_{x_i} z \quad (1 \le i \le n), \tag{34}$$

$$r_{x_i x_j} = \zeta^4 z_{x_i x_j} + 4\zeta^3 (\zeta_{x_j} z_{x_i} + \zeta_{x_i} z_{x_j}) + (4\zeta^3 \zeta_{x_i})_{x_j} z. \tag{35}$$

Assume r attains its positive maximum over $\bar{U}$ at a point $x_0 \in U$. Then

$$Dr = 0, \; D^2 r \le 0 \text{ at } x_0.$$

Thus at the point x_0,

$$\begin{aligned} 0 &\le -h_{ij} r_{x_i x_j} = \zeta^4(-h_{ij} z_{x_i x_j}) - 8\zeta^3 h_{ij} \zeta_{x_i} z_{x_j} + C\zeta^2 z \\ &\le -2\zeta^4 z^2 + 2\zeta^4 w_{x_i} z_{x_i} - 8\zeta^3 h_{ij} \zeta_{x_i} z_{x_j} + C\zeta^2 z\,, \end{aligned}$$

according to (33). Now, since $Dr = 0$ at x_0, we deduce from (34) that

$$\zeta z_{x_i} = -4 \zeta_{x_i} z \quad (1 \le i \le n).$$

Substituting above, we compute

$$\begin{aligned} 2\zeta^4 z^2 &\leq -8\zeta^3 \zeta_{x_i} w_{x_i} z + 32\zeta^2 h_{ij}\zeta_{x_i}\zeta_{x_j} z + C\zeta^2 z \\ &\leq C\zeta^3 |Dw| z + C\zeta^2 z \\ &= C\zeta^3 z^{3/2} + C\zeta^2 z. \end{aligned}$$

Finally we employ Young's inequality in the form

$$ab \leq \varepsilon a^p + C(\varepsilon) b^q \quad (a, b > 0,\ \frac{1}{p} + \frac{1}{q} = 1),$$

with $p = q = 2$ and $p = \frac{4}{3}$, $q = 4$, to deduce

$$2\zeta^4 z^2 \leq C\varepsilon\zeta^4 z^2 + C(\varepsilon).$$

Fix $\varepsilon > 0$ small enough to conclude

$$\zeta^4 z^2(x_0) \leq C.$$

Since $r = \zeta^4 z$ attains its maximum over $\bar{U}$ at x_0, we deduce that

$$\max_{\bar{U}} \zeta^4 z \leq C,$$

the constant C depending only on n and ζ. Given any region $V \subset\subset U$, we may select $\zeta \equiv 1$ on V, thereby concluding

$$\max_V z \leq C(V). \tag{36}$$

As $z = |Dw|^2 = |D(\log v)|^2$, we deduce the Harnack inequality from (36) as in the proof of Theorem 2.1.

5. If it is not true that $v = |Du| > 0$ everywhere in U, apply the above reasoning to $\tilde{u}(x) = u(x) + \varepsilon x_{n+1}$, and send $\varepsilon \to 0$ to deduce $|Du| = 0$ everywhere in U. □

Remark. The expression $z = |A|^2$ has the following geometric interpretation (cf. Aronsson [2]). Given a smooth solution u of $\Delta_\infty u = 0$, with $|Du| > 0$, set as above $\nu = Du/|Du|$ to denote the field of normals to the level sets of u. Consider then the ODE

$$\dot{x}(s) = \nu(x(s)) \quad (s \in \mathbb{R}), \tag{37}$$

whose trajectories are curves in U normal to the level sets. Then $v = |Du|$ is constant along each such curve, according to (13). The curvature is

$$\kappa = \left| \frac{d}{ds} \nu(x(s)) \right| = |D\nu \cdot \nu|.$$

But

$$\nu^i_{x_j} \nu^j = \frac{1}{v} g_{ik} u_{x_k x_j} \nu^j = \frac{1}{v} g_{ik} v_{x_k} = \frac{v_{x_i}}{v} = w_{x_i}\,.$$

Thus $\kappa = |A|$. Theorem 3.1 in particular asserts that the curvatures of each normal curve are bounded in each region $V \subset\subset U$. □

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Addenda

March 9, 1994. I should have referenced as well the interesting paper
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