



## POSITIVE ENTRIES OF STABLE MATRICES\*

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**Abstract.** The question of how many elements of a real positive stable matrix must be positive is investigated. It is shown that any real stable matrix of order greater than 1 has at least two positive entries. Furthermore, for every stable spectrum of cardinality greater than 1 there exists a real matrix with that spectrum with exactly two positive elements, where all other elements of the matrix can be chosen to be negative.

**Key words.** Stable matrix, Companion matrix, Positive elementary symmetric functions.

**AMS subject classifications.** 15A18, 15A29.

**1. Introduction.** For a square complex matrix  $A$  let  $\sigma(A)$  be the spectrum of  $A$ , that is, the set of eigenvalues of  $A$  listed with their multiplicities. Recall that a (multi) set of complex numbers is called (*positive*) *stable* if all the elements of the set have positive real parts, and that a square complex matrix  $A$  is called *stable* if  $\sigma(A)$  is stable. In this paper we investigate the question of how many elements of a real stable matrix must be positive.

We first show that a stable real matrix  $A$  has either positive diagonal elements or it has at least one positive diagonal element and one positive off-diagonal element. We then show that for any stable  $n$ -tuple  $\zeta$  of complex numbers,  $n > 1$ , such that  $\zeta$  is symmetric with respect to the real axis, there exists a real stable  $n \times n$  matrix  $A$  with exactly two positive entries such that  $\sigma(A) = \zeta$ .

The stable  $n \times n$  matrix with exactly two positive entries, whose existence is proven in Section 2, has  $(n - 1)^2$  zeros in it. In Section 3 we prove that for any stable  $n$ -tuple  $\zeta$  of complex numbers,  $n > 1$ , such that  $\zeta$  is symmetric with respect to the real axis, there exists a real stable  $n \times n$  matrix  $A$  with two positive entries and all other entries negative such that  $\sigma(A) = \zeta$ .

In Section 4 we suggest some alternative approaches to obtain the results of Section 2.

**2. Positive entries of stable matrices.** Our aim in this section is to show that for any stable  $n$ -tuple  $\zeta$  of complex numbers,  $n > 1$ , consisting of real numbers and conjugate pairs, there exists a real stable  $n \times n$  matrix  $A$  with exactly two positive entries such that  $\sigma(A) = \zeta$ . We shall first show that every real stable matrix of order

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greater than 1 has at least two positive elements. In fact we show more than that, that is, that for a stable real matrix  $A$  either all diagonal elements of  $A$  are positive or  $A$  must have at least one positive entry on the main diagonal and one off the main diagonal.

NOTATION 2.1. For an  $n$ -tuple  $\zeta = \{\zeta_1, \dots, \zeta_n\}$  of complex numbers we denote by  $s_1(\zeta), \dots, s_n(\zeta)$  the elementary symmetric functions of  $\zeta$ , that is,

$$s_k(\zeta) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \zeta_{i_1} \cdot \dots \cdot \zeta_{i_k}, \quad k = 1, \dots, n.$$

Also, we let  $s_0(\zeta) = 1$  and  $s_k(\zeta) = 0$  whenever  $k > n$  or  $k < 0$ . We say that  $\zeta$  has positive elementary symmetric functions whenever  $s_k(\zeta) > 0$ ,  $k = 1, \dots, n$ .

LEMMA 2.2. *Let  $\zeta = \{\zeta_1, \dots, \zeta_n\}$  be an  $n$ -tuple of complex numbers with positive elementary symmetric functions. Then  $\zeta$  contains no nonpositive real numbers.*

*Proof.* Note that  $\zeta$  has positive elementary symmetric functions if and only if the polynomial  $p(x) = \prod_{i=1}^n (x + \zeta_i)$  has positive coefficients. It follows that  $p(x)$  cannot have nonnegative roots, implying that none of the  $\zeta_i$ 's is a nonpositive real number.  $\square$

NOTATION 2.3. For  $\mathbb{F} = \mathbb{R}, \mathbb{C}$ , the fields of real and complex numbers respectively, we denote by  $M_n(\mathbb{F})$  the algebra of  $n \times n$  matrices with entries in  $\mathbb{F}$ . For  $A = (a_{ij})_1^n \in M_n(\mathbb{F})$  we denote by  $\text{tr } A$  the trace of  $A$ , that is, the sum  $\sum_{i=1}^n a_{ii}$ .

PROPOSITION 2.4. *Let  $A = (a_{ij})_1^n \in M_n(\mathbb{R})$ , and assume that  $\sigma(A)$  has positive elementary symmetric functions. Then either all the diagonal elements of  $A$  are positive or  $A$  has at least one positive diagonal element and one positive off-diagonal element.*

*Proof.* As is well known, the trace of  $A$  is equal to  $s_1(\sigma(A))$ , and so we have  $\sum_{i=1}^n a_{ii} > 0$ , and it follows that at least one diagonal element of  $A$  is positive. Assume that all off-diagonal elements of  $A$  are nonpositive. Such a real matrix is called a *Z-matrix*. Since the elementary symmetric functions of  $\sigma(A)$  are positive, it follows by Lemma 2.2 that  $A$  has no nonpositive real eigenvalues. Since a *Z-matrix* has no nonpositive real eigenvalues if and only if all its principal minors are positive, e.g., [1, Theorem (6.2.3), page 134], it follows that all the diagonal elements of  $A$  are positive.  $\square$

NOTATION 2.5. For an  $n$ -tuple  $\zeta = \{\zeta_1, \dots, \zeta_n\}$  of complex numbers we denote by  $\bar{\zeta}$  be the  $n$ -tuple  $\{\bar{\zeta}_1, \dots, \bar{\zeta}_n\}$ . We say that  $\bar{\zeta} = \zeta$  whenever the two  $n$ -tuples  $\bar{\zeta}$  and  $\zeta$  are identical sets.

Note that  $\bar{\bar{\zeta}} = \zeta$  if and only if all elementary symmetric functions of  $\zeta$  are real.

The following result is well known, and we provide a proof for the sake of completeness.

PROPOSITION 2.6. *Let  $\zeta$  be a stable  $n$ -tuple of complex numbers such that  $\bar{\zeta} = \zeta$ . Then  $\zeta$  has positive elementary symmetric functions.*

*Proof.* We prove our claim by induction on  $n$ . For  $n = 1, 2$  the result is trivial. Assume that the result holds for  $n \leq m$  where  $m \geq 2$ , and let  $n = m + 1$ . Assume first that  $\zeta$  contains a positive number  $\lambda$ , and let  $\zeta'$  be the  $(n - 1)$ -tuple obtained by eliminating  $\lambda$  from  $\zeta$ . Note that  $\zeta'$  is stable and  $\overline{\zeta'} = \zeta'$ . By the inductive assumption we have  $s_k(\zeta') > 0$ ,  $k = 1, \dots, n - 1$ , and it follows that

$$s_k(\zeta) = s_k(\zeta') + \lambda s_{k-1}(\zeta') > 0, \quad k = 1, \dots, n.$$

If  $\zeta$  does not contain a positive number then it contains a conjugate pair  $\{\lambda, \bar{\lambda}\}$ , where  $\text{Re}(\lambda) > 0$ . Let  $\zeta''$  be the  $(n - 2)$ -tuple obtained by eliminating  $\lambda$  and  $\bar{\lambda}$  from  $\zeta$ . Note that the  $\zeta''$  is stable and  $\overline{\zeta''} = \zeta''$ . By the inductive assumption we have  $s_k(\zeta'') > 0$ ,  $k = 1, \dots, n - 2$ , and it follows that

$$s_k(\zeta) = s_k(\zeta'') + 2\text{Re}(\lambda)s_{k-1}(\zeta'') > 0 + |\lambda|^2 s_{k-2}(\zeta'') > 0, \quad k = 1, \dots, n,$$

proving our claim.  $\square$

It is easy to show that the converse of Proposition 2.6 holds when  $n \leq 2$ . However, the converse does not hold for a larger  $n$ , as is demonstrated by the nonstable triple  $\zeta = \{3, -1 + 3i, -1 - 3i\}$ , whose elementary symmetric functions are positive.

As a corollary of Propositions 2.4 and 2.6 we obtain:

**COROLLARY 2.7.** *Let  $A$  be a stable real square matrix. Then either all the diagonal elements of  $A$  are positive or  $A$  has at least one positive diagonal element and one positive off-diagonal element.*

In order to prove the existence of a real stable  $n \times n$  matrix  $A$  with exactly two positive entries, we introduce:

**NOTATION 2.8.** Let  $n$  be a positive integer. For an  $n$ -tuple  $\zeta$  of complex numbers we denote by  $C_1(\zeta)$ ,  $C_2(\zeta)$  and  $C_3(\zeta)$  the matrices

$$C_1(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & (-1)^{n-1} s_n(\zeta) \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & (-1)^{n-2} s_{n-1}(\zeta) \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & (-1)^{n-3} s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & -s_2(\zeta) \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & s_1(\zeta) \end{pmatrix},$$

$$C_2(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & s_n(\zeta) \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & s_{n-1}(\zeta) \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & s_2(\zeta) \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & s_1(\zeta) \end{pmatrix},$$

$$C_3(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & -s_n(\zeta) \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & -s_{n-1}(\zeta) \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & -s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & -s_2(\zeta) \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & s_1(\zeta) \end{pmatrix}.$$

Recall that  $A \in M_n(\mathbb{C})$  is called nonderogatory if for every eigenvalue  $\lambda$  of  $A$  the Jordan canonical form of  $A$  has exactly one Jordan block corresponding to  $\lambda$ . Equivalently, the minimal polynomial of  $A$  is equal to the characteristic polynomial of  $A$ .

LEMMA 2.9. *Let  $n$  be a positive integer,  $n > 1$ , and let  $\zeta = \{\zeta_1, \dots, \zeta_n\}$  be an  $n$ -tuple of complex numbers. Then the matrices  $C_1(\zeta)$ ,  $C_2(\zeta)$  and  $C_3(\zeta)$  are diagonally similar, are nonderogatory and share the spectrum  $\zeta$ .*

*Proof.* The matrix  $C_1(\zeta)$  is the companion matrix of the polynomial  $q(x) = \prod_{i=1}^n (x - \zeta_i)$ . Hence  $\sigma(C_1(\zeta)) = \zeta$  and  $C_1(\zeta)$  is nonderogatory. Clearly

$$C_2(\zeta) = D_1 C_1(\zeta) D_1, \quad \text{where } D_1 = \text{diag}((-1)^1, (-1)^2, \dots, (-1)^n),$$

and

$$C_3(\zeta) = D_2 C_2(\zeta) D_2, \quad \text{where } D_2 = \text{diag}(1, 1, \dots, 1, -1).$$

Our claim follows.  $\square$

In view of Lemma 2.9, the claim of Proposition 2.6 on  $C_3(\zeta)$  yields the following main result of this section.

THEOREM 2.10. *Let  $n$  be a positive integer,  $n > 1$ , and let  $\zeta$  be an  $n$ -tuple of complex numbers such that  $\bar{\zeta} = \zeta$ . If  $\zeta$  has positive elementary symmetric functions then there exists a matrix  $A \in M_n(\mathbb{R})$  such that  $\sigma(A) = \zeta$  and  $A$  has one positive diagonal entry and one positive off-diagonal entry, while all other entries of  $A$  are nonpositive. In particular, every nonderogatory stable matrix  $A \in M_n(\mathbb{R})$  is similar to a real  $n \times n$  matrix which has exactly two positive entries.*

**3. Eliminating the zero entries.** The proof of Theorem 2.10 uses the matrix  $C_3(\zeta)$  which has  $(n-1)^2$  zero entries. The aim of this section is to strengthen Theorem 2.10 by replacing  $C_3(\zeta)$  with a real matrix  $A$ , having exactly two positive entries, all other entries being negative and  $\sigma(A) = \zeta$ .

We start with a weaker result, which one gets easily using perturbation techniques. Let  $A \in M_n(\mathbb{R})$  and let  $\|\cdot\| : M_n(\mathbb{R}) \rightarrow [0, \infty)$  be the  $l_2$  operator norm. Since the eigenvalues of a  $A$  depend continuously on the entries of the  $A$ , it follows that if  $\sigma(A)$  has positive elementary symmetric functions, then for  $\varepsilon > 0$  sufficiently small, every matrix  $\tilde{A} \in M_n(\mathbb{R})$  with  $\|\tilde{A} - A\| < \varepsilon$  has a spectrum  $\sigma(\tilde{A})$  with positive elementary symmetric functions. Also, if  $A$  is stable then for  $\varepsilon > 0$  sufficiently small, every matrix

$\tilde{A} \in M_n(\mathbb{R})$  with  $\|\tilde{A} - A\| < \varepsilon$  is stable. Consequently, it follows immediately from Theorem 2.10 that

**COROLLARY 3.1.** *For a positive integer  $n$ ,  $n > 1$ , there exists a matrix  $A \in M_n(\mathbb{R})$  such that  $\sigma(A)$  has positive elementary symmetric functions and  $A$  has one positive diagonal entry and one positive off-diagonal entry, while all other entries of  $A$  are negative. Furthermore, the matrix  $A$  can be chosen to be stable.*

In the rest of this section we prove that one can find such a matrix  $A$  with any prescribed stable spectrum.

**LEMMA 3.2.** *Let  $n$  be a positive integer,  $n > 1$ , let  $\zeta$  be an  $n$ -tuple of complex numbers such that  $\bar{\zeta} = \zeta$ , and assume that  $\zeta$  has positive elementary symmetric functions. Suppose that there exists  $X \in M_n(\mathbb{R})$  such that*

$$(C_3(\zeta))_{ij} = 0 \implies (C_3(\zeta)X - XC_3(\zeta))_{ij} < 0, \quad i, j = 1, \dots, n.$$

*Then there exists  $A \in M_n(\mathbb{R})$  similar to  $C_3(\zeta)$  such that  $a_{nn}, a_{n,n-1} > 0$  and all other entries of  $A$  are negative.*

*Proof.* Assume the existence of such a matrix  $X$ . Define the matrix  $T(t) = I - tX$ ,  $t \in \mathbb{R}$ . Let  $r = \|X\|^{-1}$ . Using the Neumann series expansion, e.g., [2, page 7], for  $|t| < r$  we have  $T(t)^{-1} = \sum_{i=0}^{\infty} t^i X^i$ . The matrix  $A(t) = T(t)C_3(\zeta)T(t)^{-1}$  thus satisfies

$$A(t) = C_3(\zeta) + t(C_3(\zeta)X - XC_3(\zeta)) + O(t^2).$$

Therefore, there exists  $\varepsilon \in (0, r)$  such that for  $t \in (0, \varepsilon)$  the matrix  $A(t)$  has positive entries in the  $(n, n-1)$  and  $(n, n)$  positions, while all other entries of  $A(t)$  are negative.  $\square$

The following lemma is well known, and we provide a proof for the sake of completeness.

**LEMMA 3.3.** *Let  $A, B \in M_n(\mathbb{F})$  where  $\mathbb{F}$  is  $\mathbb{R}$  or  $\mathbb{C}$ . The following are equivalent.*

- (i) *The system  $AX - XA = B$  is solvable over  $\mathbb{F}$ .*
- (ii) *For every matrix  $E \in M_n(\mathbb{F})$  that commutes with  $A$  we have  $\text{tr} BE = 0$ .*

*Proof.* (i) $\implies$ (ii). Let  $E \in M_n(\mathbb{F})$  commute with  $A$ . Then

$$\text{tr} BE = \text{tr}(AX - XA)E = \text{tr} AXE - \text{tr} XEA = \text{tr} XEA - \text{tr} XEA = 0.$$

(ii) $\implies$ (i). Consider the linear operator  $L : M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$  defined by  $L(X) = AX - XA$ . Its kernel consists of all matrices in  $M_n(\mathbb{F})$  commuting with  $A$ . By the previous implication, the image of  $L$  is contained in the subspace  $V$  of  $M_n(\mathbb{F})$  consisting of all matrices  $C$  such that  $\text{tr} CE = 0$  whenever  $E \in \text{kernel}(L)$ . Since clearly  $\dim(V) = n^2 - \dim(\text{kernel}(L)) = \dim(\text{image}(L))$ , it follows that  $\text{image}(L) = V$ .  $\square$

**THEOREM 3.4.** *Let  $n$  be a positive integer,  $n > 1$ , and let  $\zeta$  be an  $n$ -tuple of complex numbers. Let  $b_{ij}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, n-1$  be given complex numbers, and*



let  $C = C_k(\zeta)$  for some  $k \in \{1, 2, 3\}$ . Then there exists unique  $b_{in} \in \mathbb{C}$ ,  $i = 1, \dots, n$ , such that for the matrix  $B = (b_{ij})_1^n \in M_n(\mathbb{C})$  the system  $CX - XC = B$  is solvable. Furthermore, if  $\bar{\zeta} = \zeta$  and  $b_{ij}$  is real for  $i = 1, \dots, n$ ,  $j = 1, \dots, n-1$ , then the matrix  $B$  is real, and the solution  $X$  can be chosen to be real.

*Proof.* Since  $C_2(\zeta)$  and  $C_3(\zeta)$  are diagonally similar to  $C_1(\zeta)$ , where the corresponding diagonal matrices are real, it is enough to prove the theorem for  $C = C_1(\zeta)$ . So, let  $C = C_1(\zeta)$  and consider the system

$$CX - XC = B. \tag{3.5}$$

As is well known, e.g., [3, Corollary 1, page 222], since  $C = C_1(\zeta)$  is nonderogatory, every matrix that commutes with  $C$  is a polynomial in  $C$ . Therefore, it follows from Lemma 3.3 that the system (3.5) is solvable if and only if

$$\text{tr } BC^k = 0, \quad k = 0, \dots, n-1. \tag{3.6}$$

Denote  $v_k = b_{n+1-k,n}$ ,  $k = 1, \dots, n$ . Note that (3.6) is a system of  $n$  equations in the variables  $v_1, \dots, v_n$ . Furthermore, it is easy to verify that the first nonzero element in the  $n$ th row of  $C^k$  is located at the position  $(n, n-k)$  and its value is 1. It follows that if we write (3.6) as  $Ev = f$ , where  $E \in M_n(\mathbb{C})$  and  $v = (v_1, \dots, v_n)^T$ , then  $E$  is a lower triangular matrix with 1's along the main diagonal. It follows that the matrix  $B$  is uniquely determined by (3.6).

If  $\bar{\zeta} = \zeta$  and  $b_{ij}$  is real for  $i = 1, \dots, n$ ,  $j = 1, \dots, n-1$  then  $C = C_1(\zeta)$  is real and hence the system (3.6) has real coefficients, and the uniquely determined  $B$  is real. It follows that the system (3.5) is real, and so it has a real solution  $X$ .  $\square$

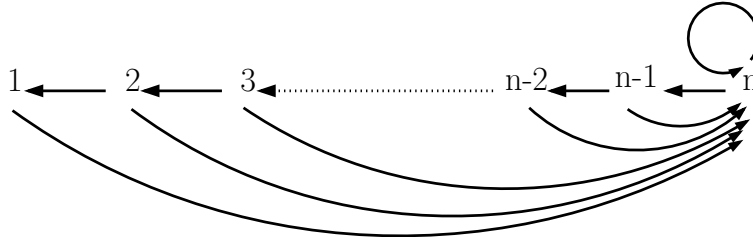
If we choose the numbers  $b_{ij}$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, n-1$ , to be negative, then Lemma 3.2 and Theorem 3.4 yield

**COROLLARY 3.7.** *Let  $n$  be a positive integer,  $n > 1$ , let  $\zeta$  be an  $n$ -tuple of complex numbers, and assume that the elementary symmetric functions of  $\zeta$  are positive. Then there exists a matrix  $A \in M_n(\mathbb{R})$  with  $\sigma(A) = \zeta$  such that  $A$  has one positive diagonal element, one positive off-diagonal element and all other entries of  $A$  are negative. In particular, the above holds for stable  $n$ -tuples  $\zeta$  such that  $\bar{\zeta} = \zeta$ .*

**4. Other types of companion matrices.** Another way to prove some of the results of Section 2 is to parameterize the companion matrices in Notation 2.8. Consider

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_0 \\ \beta_0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_1 \\ 0 & \beta_1 & 0 & \cdots & 0 & 0 & 0 & \gamma_2 \\ & & & \dots & & & & \\ 0 & 0 & 0 & \cdots & 0 & \beta_{n-3} & 0 & \gamma_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-2} & \gamma_{n-1} \end{pmatrix}.$$

Looking at the directed graph of  $C$ , which is



one can immediately see that there is exactly one simple cycle of length  $k$  for  $1 \leq k \leq n$ , that is,  $(n, \dots, n+1-k)$ . Therefore, the only nonzero principal minors of  $C$  are those whose rows and columns are indexed by  $\{k, \dots, n\}$ ,  $k = 1, \dots, n$ , and their respective values are  $(-1)^{n-k} \gamma_{k-1} \beta_{k-1} \cdots \beta_{n-2}$  for  $k < n$  and  $\gamma_{n-1}$  for  $k = n$ . It follows that the characteristic polynomial  $\chi_C(x)$  of  $C$  is

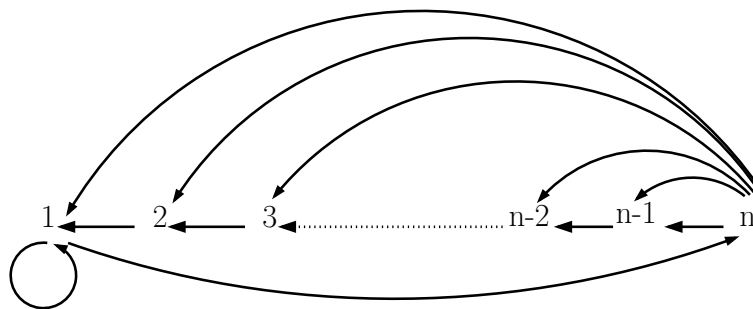
$$\chi_C(x) = x^n - \gamma_{n-1}x^{n-1} - \gamma_{n-2}\beta_{n-2}x^{n-2} - \gamma_{n-3}\beta_{n-3}\beta_{n-2}x^{n-3} - \dots - \gamma_1\beta_1\beta_2 \cdots \beta_{n-2}x - \gamma_0\beta_0\beta_1 \cdots \beta_{n-2}. \quad (4.1)$$

Using this explicit formula, one can prove directly the claim contained in Lemma 2.9 that the matrices  $C_1(\zeta)$ ,  $C_2(\zeta)$  and  $C_3(\zeta)$  share the spectrum  $\zeta$ .

There are other possibilities to generate companion matrices. For example, consider the matrix

$$L = \begin{pmatrix} \gamma_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_{n-2} \\ \beta_{n-3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \beta_{n-4} & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & \dots & & & & \\ 0 & 0 & 0 & \cdots & 0 & \beta_0 & 0 & 0 \\ \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0 \end{pmatrix}.$$

The directed graph of  $L$  is



Again it is clear that there is exactly one simple cycle of length  $k$  for any  $1 \leq k \leq n$ , that is,  $(1)$  for  $k = 1$  and  $(n, k-1, \dots, 1)$  for  $1 < k \leq n$ . Therefore, the only nonzero  $1 \times 1$  principal minor of  $L$  is  $l_{11} = \gamma_{n-1}$ , and for  $1 < k \leq n$  the only

nonzero  $k \times k$  principal minor of  $L$  is the one whose rows and columns are indexed by  $\{1, \dots, k-1, n\}$ , and its value is  $(-1)^{n-k} \gamma_{k-1} \beta_{k-1} \cdots \beta_{n-2}$ . It follows that the characteristic polynomial  $\chi_L(x)$  of  $L$  is identical to  $\chi_C(x)$ . Note that there is no permutation matrix  $P$  with  $P^T C P = L$  or  $P^T C^T P = L$ .

Now, take the following specific choice of the parameters  $\beta$  and  $\gamma$ :

$$L_1 = \begin{pmatrix} -p_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & \dots & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ -p_{n-2} & -p_{n-3} & -p_{n-4} & \cdots & -p_2 & -p_1 & -p_0 & 0 \end{pmatrix}.$$

By (4.1), the characteristic polynomial computes to

$$\chi_{L_1}(x) = \sum_{\nu=0}^n p_\nu x^\nu,$$

where  $p_n = 1$ .

So  $L_1$  is another kind of companion matrix. Note that  $L_1$  is almost lower triangular, with only one nonzero element above the main diagonal and one on the main diagonal.

Another specific choice of the parameters  $\beta$  and  $\gamma$  can be used to produce another direct proof of Theorem 2.10. For an  $n$ -tuple  $\zeta$  of complex numbers with  $\zeta = \bar{\zeta}$  and positive elementary symmetric functions, the polynomial  $q(x) = \prod_{i=1}^n (x - \zeta_i) = \sum_{i=0}^n q_i x^i$  has coefficients  $q_i$ ,  $0 \leq i \leq n$  of alternating signs, where  $q_n = 1$ . By (4.1), the polynomial  $q(x)$  is the characteristic polynomial of the matrix

$$\begin{pmatrix} -q_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & \dots & & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ -q_{n-2} & q_{n-3} & -q_{n-4} & \cdots & (-1)^{n-3} q_2 & (-1)^{n-2} q_1 & (-1)^{n-1} q_0 & 0 \end{pmatrix},$$

which has exactly two positive entries, that is,  $-q_{n-1}$  on the diagonal and 1 in the right upper corner.

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