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A BRUHAT ORDER FOR THE CLASS OF $(0,1)$-MATRICES WITH ROW SUM VECTOR R AND COLUMN SUM VECTOR S*

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Abstract. Generalizing the Bruhat order for permutations (so for permutation matrices), a Bruhat order is defined for the class of m by n $(0,1)$-matrices with a given row and column sum vector. An algorithm is given for constructing a minimal matrix (with respect to the Bruhat order) in such a class. This algorithm simplifies in the case that the row and column sums are all equal to a constant k. When $k=2$ or $k=3$, all minimal matrices are determined. Examples are presented that suggest such a determination might be very difficult for $k \geq 4$.

Key words. Bruhat order, Row sum and column sum vectors, Interchanges, Minimal matrix.

AMS subject classifications. 05B20, 06A07, 15A36.

1. Introduction.

Let $R=(r_1,r_2,\ldots,r_m)$ and $S=(s_1,s_2,\ldots,s_n)$ be nonincreasing, positive integral vectors, so that

$$r_1 \geq r_2 \geq \cdots \geq r_m > 0 \quad \text{and} \quad s_1 \geq s_2 \geq \cdots \geq s_n > 0. \tag{1.1}$$

Then $\mathcal{A}(R,S)$ denotes the class of all m by n $(0,1)$-matrices with row sum vector R and column sum vector S.

The row and column sum vectors R and S of a $(0,1)$- matrix are partitions of the same integer t (its number of 1's). Let $R^*=(r_1^*,r_2^*,\ldots,r_n^*)$ denote the *conjugate* of R (with trailing 0's included to get an n-tuple). The class $\mathcal{A}(R,R^*)$ is nonempty, and it contains a unique matrix, the *perfectly nested matrix* $\overline{A}$ with all 1's left justified. Let R and S be proposed row and column sum monotone vectors of a $(0,1)$-matrix that satisfy (1.1). The Gale-Ryser Theorem (see e.g., [4]) asserts that $\mathcal{A}(R,S)$ is nonempty if and only if S is *majorized* by R^* (written $S \preceq R^*$), that is,

$$s_1+\cdots+s_k \leq r_1^*+\cdots+r_k^* \quad (k=1,2,\ldots,n)$$

with equality for $k=n$. If $\mathcal{A}(R,S) \neq \emptyset$, then every matrix in $\mathcal{A}(R,S)$ can be obtained from the perfectly nested matrix $\overline{A}$ with row and column sum vectors R and R^*, respectively, by shifting 1's in rows to the right. Ryser also proved that given matrices A and B in $\mathcal{A}(R,S)$ then B can be gotten from A by a sequence of *interchanges*

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftrightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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which replace a submatrix equal to L_2 by I_2, or the other way around.

There is a well-known order on the symmetric group S_n (more generally, on Coxeter groups) of permutations of $\{1,2,\ldots,n\}$ called the *Bruhat order*, given by:

If τ and π are permutations, then $\pi \leq_B \tau$ (in the Bruhat order) provided π can be gotten from τ by a sequence of transformations of the form:

If $a_i > a_j$, then $a_1 \cdots a_i \cdots a_j \cdots a_n$ is replaced with $a_1 \cdots a_j \cdots a_i \cdots a_n$.

Thus if $n = 3$, 123 is the unique minimal element and 321 is the unique maximal element in the Bruhat order on S_3.

As usual, the permutations in S_n can be identified with the permutation matrices of order n, where the permutation τ corresponds to the permutation matrix $P = [p_{ij}]$ with $p_{ij} = 1$ if and only if $j = \tau(i)$. If P and Q are permutation matrices of order n corresponding to permutations τ and π, then we write $P \leq_B Q$ whenever $\tau \leq_B \pi$. The reduction in the Bruhat order, interpreted for permutation matrices, is that of *one-sided interchanges*:

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \to I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For $n = 3$, the minimal permutation (matrix) in the Bruhat order is

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the maximal permutation matrix is

$$D_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

There are equivalent ways to define the Bruhat order on S_n. One is in terms of the *Gale order* (see e.g., [1]) on subsets of size k of $\{1,2,\ldots,n\}$. Let k be an integer with $1 \leq k \leq n$, and let $X = \{a_1, a_2, \ldots, a_k\}$ and $Y = \{b_1, b_2, \ldots, b_k\}$ be subsets of $\{1,2,\ldots,n\}$ of size k where $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$. Then in the Gale order, $X \leq_G Y$ if and only if $a_1 \leq b_1, a_2 \leq b_2, \ldots, a_k \leq b_k$. For $\tau = i_1 i_2 \ldots i_n \in S_n$, let $\tau[k] = \{i_1, i_2, \ldots, i_k\}$. Then it is straightforward to check that, if also $\pi \in S_n$, then

$$\tau \leq_B \pi \quad \text{ if and only if } \quad \tau[k] \leq_G \pi[k] \quad (k = 1, 2, \ldots, n).$$

For an m by n matrix $A = [a_{ij}]$, let Σ_A denote the m by n matrix whose (k,l)-entry equals

$$\sigma_{kl}(A) = \sum_{i=1}^{k} \sum_{j=1}^{l} a_{ij} \quad (1 \leq k \leq m; 1 \leq l \leq n),$$

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the sum of the entries in the leading k by l submatrix of A. Using the Gale order, one easily checks that for permutation matrices P and Q of order n, $P \leq_B Q$ if and only if $\Sigma_P \geq \Sigma_Q$, where this latter order is *entrywise order.*

The Bruhat order on permutation matrices can be extended to the classes $\mathcal{A}(R,S)$. For A_1 and A_2 in $\mathcal{A}(R,S)$ we define $A_1 \leq_B A_2$ provided, in the entrywise order, $\Sigma_{A_1} \geq \Sigma_{A_2}$. It is immediate that if A_1 and A_2 are matrices in $\mathcal{A}(R,S)$ and A_1 is obtained from A_2 by a sequence of one-sided interchanges

$$L_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then $A_1 \leq_B A_2$. This observation gives the following corollary.

COROLLARY 1.1. *Let A be a matrix in $\mathcal{A}(R,S)$ that is minimal in the Bruhat order. Then no submatrix of A equals L_2.*

EXAMPLE. Let $R = S = (2,2,2,2,2)$. Then

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Sigma_A = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 5 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Sigma_B = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 & 6 \\ 2 & 4 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 & 10 \end{bmatrix}$$

are both minimal elements of $\mathcal{A}(R,S)$ in the Bruhat order.

Let A be a matrix in $\mathcal{A}(R,S)$ which is minimal in the Bruhat order. Let $A^c = J_{m,n} - A$ be the *complement* of A. Here $J_{m,n}$ is the m by n matrix of all 1's (abbreviated to J_n when $m = n$), and thus A^c has 1's exactly where A has 0's. Let R^c and S^c be, respectively, the row and column sum vectors of A^c. Since R and S are monotone nonincreasing, R^c and S^c are monotone nondecreasing. Since $\Sigma_{A^c} = \Sigma_{J_{m,n}} - \Sigma_A$, it follows that, after reordering rows and columns to get monotone nonincreasing vectors $\widehat{R^c} = (n - r_m, \ldots, n - r_1)$ and $\widehat{S^c} = (m - s_n, \ldots, m - s_1)$, the resulting matrix $\widehat{A^c}$ is a maximal matrix in the class $\mathcal{A}(\widehat{R^c}, \widehat{S^c})$.

EXAMPLE. Let $R = S = (2,2,2,2,2)$. Then $\widehat{R^c} = \widehat{S^c} = (3,3,3,3,3)$. A matrix in $\mathcal{A}(\widehat{R^c}, \widehat{S^c})$ that is minimal in the Bruhat order is the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

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Thus the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

is a matrix in $\mathcal{A}(R,S)$ that is maximal in the Bruhat order.

2. An Algorithm for a Minimal Matrix. In this section we give an algorithm that, starting from the perfectly nested matrix in $\mathcal{A}(R,R^*)$, constructs a matrix in $\mathcal{A}(R,S)$ that is minimal in the Bruhat order. From the above discussion, it follows that we also get an algorithm for constructing a matrix in $\mathcal{A}(R,S)$ that is maximal in the Bruhat order.

I. Algorithm to Construct a Minimal Matrix in the Bruhat Order

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be monotone nonincreasing positive integral vectors with $S \preceq R^*$. Let $\overline{A}$ be the unique matrix in $\mathcal{A}(R,R^*)$.

1. Rewrite R by grouping together its components of equal value:

$$R = (a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_k, \ldots, a_k)$$

where $a_1 > a_2 > \cdots > a_k$, and the number of a_i's equals p_i, $(i = 1, 2, \ldots, k)$.

2. Determine nonnegative integers $x_1, x_2, \ldots, x_k$ satisfying $x_1 + x_2 + \cdots + x_k = s_n$ where $x_k, x_{k-1}, \ldots, x_1$ are maximized in turn *in this order* subject to $(s_1, s_2, \ldots, s_{n-1}) \preceq R_1^*$ where $R_1 = R_{(x_1,x_2,\ldots,x_k)}$ is the vector

$$(\underbrace{a_1, \ldots, a_1, \overbrace{a_1 - 1, \ldots, a_1 - 1}^{x_1}}_{p_1}, \ldots, \underbrace{a_k, \ldots, a_k\, \overbrace{a_k - 1, \ldots, a_k - 1}^{x_k}}_{p_k}).$$

3. Shift $s_n = x_1 + x_2 + \cdots + x_k$ 1's to the last column as specified by those rows whose sums have been diminished by 1: thus the last column consists of $p_1 - x_1$ 0's followed by x_1 1's, …, $p_k - x_k$ 0's followed by x_k 1's.
4. Proceed recursively and return to step 1, with R replaced with R_1 and S replaced with $S_1 = (s_1, s_2, \ldots, s_{n-1})$

EXAMPLE. Let $R = (4, 4, 3, 3, 2, 2), S = (4, 4, 3, 3, 3, 1)$. Then $R^* = (6, 6, 4, 2, 0, 0)$. Starting with the matrix $\overline{A}$ in $\mathcal{A}(R,R^*)$ and applying the algorithm, we get:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$

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$$\begin{bmatrix} 1&1&1&1&0&0\\ 1&1&1&1&0&0\\ 1&1&1&0&0&0\\ 1&1&0&0&1&0\\ 1&0&0&0&1&0\\ 0&0&0&0&1&1 \end{bmatrix} \rightarrow \begin{bmatrix} 1&1&1&1&0&0\\ 1&1&1&1&0&0\\ 1&1&1&0&0&0\\ 1&1&0&0&1&0\\ 0&0&0&1&1&0\\ 0&0&0&0&1&1 \end{bmatrix}.$$

We can stop at this point since no more shifting has to be done. The resulting matrix has no submatrix equal to L_2, and it is straightforward to verify that it is a minimal matrix in its class $\mathcal{A}(R, S)$.

THEOREM 2.1. *Let R and S be positive, monotone vectors such that $\mathcal{A}(R, S)$ is nonempty. Then algorithm I constructs a matrix $A = [a_{ij}]$ in $\mathcal{A}(R, S)$ that is minimal in the Bruhat order.*

Proof. We prove the theorem by induction on n. If $n = 1$, there is a unique matrix in $\mathcal{A}(R, S)$, and the theorem holds trivially. Assume that $n > 1$. Let R_1 be defined as in the algorithm. Let $P = [p_{ij}]$ be a matrix in $\mathcal{A}(R, S)$ such that $P \preceq_B A$. Let $u = (u_1, u_2, \ldots, u_m)^T$ and $v = (v_1, v_2, \ldots, v_m)^T$ be, respectively, the last columns of A and P. First suppose that $u = v$. Then the matrices A' and P' obtained by deleting the last column of A and P, respectively, belong to the same class $\mathcal{A}(R', S')$, and $P' \preceq_B A'$. Since A' is constructed by algorithm I, it now follows from the inductive assumption that $P' = A'$ and hence $P = A$.

Now suppose that $u \neq v$. We may assume that the last column of P consists of $p_1 - y_1$ 0's followed by y_1 1's, $\ldots$, $p_k - y_k$ 0's followed by y_k 1's where $y_1, y_2, \ldots, y_k$ are nonnegative integers satisfying $y_1 + y_2 + \cdots + y_k = s_n$. Otherwise, the last column of P contains a 1 above a 0 in two rows with equal sums, and P contains a submatrix equal to L_2. A one-sided interchange then replaces P with Q where $Q \preceq_B P \preceq_B A$.

The row sum vector $R_{(y_1,y_2,\ldots,y_k)}$ of the matrix P' obtained by deleting the last column of P is nonincreasing. Since $P \in \mathcal{A}(R, S)$, $(s_1, s_2, \ldots, s_{n-1}) \preceq R^*_{(y_1,y_2,\ldots,y_k)}$. The choice of $x_1, x_2, \ldots, x_k$ implies that

$$y_1 + \cdots + y_j \leq x_1 + \cdots + x_j \quad (j = 1, 2, \ldots, k) \tag{2.1}$$

with equality for $j = k$. Let q be the smallest integer such that $u_q \neq v_q$. Then it follows from (2.1) that $u_q = 0$ and $v_q = 1$. We calculate that

$$\begin{aligned} \sum_{i=1}^{q} \sum_{j=1}^{n-1} p_{ij} &= r_1 + \cdots + r_q - \sum_{j=1}^{q-1} v_j - 1 \\ &= r_1 + \cdots + r_q - \sum_{j=1}^{q-1} u_j - 1 \\ &= r_1 + \cdots + r_q - \sum_{j=1}^{q} u_j - 1 \end{aligned}$$

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$$= \sum_{i=1}^{q} \sum_{j=1}^{n-1} a_{ij} - 1,$$

contradicting that $P \preceq_B A$. The theorem now follows. □

We now consider classes $\mathcal{A}$ with constant row and column sums. Let k be an integer with $1 \leq k \leq n$, let $K = (k, k, \ldots, k)$, the n-vector of k's, and let $R = S = K$. We denote the corresponding class $\mathcal{A}(R, S)$ by $\mathcal{A}(n, k)$. In case $k = 1$, this gives the class of permutation matrices of order n. Our algorithm for constructing a minimal matrix in $\mathcal{A}(K, K)$ simplifies in this case.

II. Algorithm to Construct a Minimal Matrix in the Bruhat order for $\mathcal{A}(n, k)$

1. Let $n = qk + r$ where $0 \leq r < k$.
2. If $r = 0$, then $A = J_k \oplus \cdots \oplus J_k$, ($q$ J_k's) is a minimal matrix.
3. Else, $r \neq 0$.
   (a) If $q \geq 2$, let
   $$A = X \oplus J_k \oplus \cdots \oplus J_k, \ (q-1 \ J_k\text{'s}, \ X \text{ has order } k + r),$$
   and let $n \leftarrow k + r$.
   (b) Else, $q = 1$, and let
   $$A = \left[\begin{array}{c|c} J_{r,k} & O_k \\ \hline X & J_{k,r} \end{array}\right], \ (X \text{ has order } k),$$
   and let $n \leftarrow k$ and $k \leftarrow k - r$.
   (c) Proceed recursively with the current values of n and k to determine X.

Example. Let $n = 18$ and $k = 11$. The algorithm constructs the following minimal matrix in $\mathcal{A}(K, K)$.

$$\left[\begin{array}{c|c} J_{7,11} & O_7 \\ \hline \begin{array}{c|c} \begin{array}{c|c} J_{3,4} & O_3 \\ \hline I_4 & J_{4,3} \end{array} & O_{7,4} \\ \hline O_{4,7} & J_4 \end{array} & J_{11,7} \end{array}\right].$$

Here we first construct (with $18 = 1 \cdot 11 + 7$),

$$\left[\begin{array}{c|c} J_{7,11} & O_7 \\ \hline X & J_{11,7} \end{array}\right].$$

Then to construct the matrix X of order 11 with $k = 11 - 7 = 4$ (and $11 = 2 \cdot 4 + 3$), we construct

$$\left[\begin{array}{c|c} Y & O_{7,4} \\ \hline O_{4,7} & J_4 \end{array}\right].$$

Then to construct the matrix Y of order $4 + 3 = 7$ with $k = 4$ (and $7 = 1 \cdot 4 + 3$), we construct

$$\left[\begin{array}{c|c} J_{3,4} & O_3 \\ \hline Z & J_{4,3} \end{array}\right].$$

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Finally, to construct the matrix Z of order 4 with $k = 4 - 3 = 1$ (and $4 = 4 \cdot 1 + 0$), we construct

$$Z = I_1 \oplus I_1 \oplus I_1 \oplus I_1 = I_4.$$

3. Minimal Matrices in $\mathcal{A}(n, 2)$ and $\mathcal{A}(n, 3)$. In this section we characterize the minimal matrices in the classes $\mathcal{A}(n, 2)$ and $\mathcal{A}(n, 3)$. Clearly, if A is minimal, so is its transpose A^T. We first record a useful lemma.

LEMMA 3.1. *Let k and n be positive integers with $n \geq k$, and let $A = [a_{ij}]$ be a matrix in $\mathcal{A}(n, k)$. Assume that A is minimal in the Bruhat order. Let p and q be integers with $1 \leq p < q \leq n$, and let r be an integer with $0 \leq r < n$. If*

$$a_{1p} + a_{2p} + \cdots + a_{rp} = a_{1q} + a_{2q} + \cdots + a_{rq}, \tag{3.1}$$

then $(a_{r+1,p}, a_{r+1,q}) \neq (0, 1)$. (If $r = 0$, then both sides of (3.1) *are interpreted as* 0.*)*

Proof. Assume that (3.1) holds and $(a_{r+1,p}, a_{r+1,q}) = (0, 1)$. Since A has k 1's in each column, there exists an integer s with $r + 1 < s \leq n$ such that $(a_{sp}, a_{sq}) = (1, 0)$. Hence A has a submatrix of order 2 equal to L_2, and A cannot be minimal in the Bruhat order. □

The minimal matrices in $\mathcal{A}(n, 2)$ are easily determined. Let F_n denote the matrix of order n with 0's in positions $(1, n), (2, n - 2), \ldots, (n, 1)$ and 0's elsewhere.

THEOREM 3.2. *Let n be an integer with $n \geq 2$. Then a matrix in $\mathcal{A}(n, 2)$ is a minimal matrix in the Bruhat order if and only if it is the direct sum of matrices equal to J_2 and F_3.*

Proof. Let $A = (a_{ij})$ be a minimal matrix in $\mathcal{A}(n, 2)$. It follows from several applications of Lemma 3.1 (the case $r = 0$) to A and its transpose that A has the form

$$\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & a_{22} & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}.$$

If $a_{22} = 1$, then $A = J_2 \oplus A'$ where A' is a minimal matrix in $\mathcal{A}(n - 2, 2)$. Suppose that $a_{22} = 0$. There exists $i, j \geq 3$ such that $a_{2j} = a_{i2} = 1$. Since A cannot have a submatrix equal to L_2, $a_{ij} = 1$, and then it follows that $i = j = 3$. Hence $A = F_3 \oplus A'$ where A' is a minimal matrix in $\mathcal{A}(n - 3, 2)$. The theorem now follows by induction on n. □

Let

$$V = \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right].$$

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For $i \geq 1$, let U_i be the matrix in $\mathcal{A}(i+6,3)$ of the form

$$\begin{bmatrix} 1 & 1 & 1 & 0 & & & \cdots & & \\ 1 & 1 & 1 & 0 & & & \cdots & & \\ 1 & 1 & 0 & 1 & & & \cdots & & \\ 0 & 0 & 1 & 1 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & 1 & 1 & 1 & 0 & 0 \\ & & & & & 1 & 0 & 1 & 1 \\ & & & & & 0 & 1 & 1 & 1 \\ & & & & & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Thus

$$U_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

THEOREM 3.3. *Let n be an integer with $n \geq 3$. Then a matrix in $\mathcal{A}(n,3)$ is a minimal matrix in the Bruhat order if and only if it is the direct sum of matrices equal to J_3, F_4, V, V^T and U_i $(i \geq 1)$.*

Proof. Let $A = [a_{ij}]$ be a minimal matrix in $\mathcal{A}(n,3)$. Then A has the form

$$\begin{bmatrix} 1 & 1 & 1 & 0 & \cdots & 0 \\ 1 & a_{22} & & & & \\ 1 & & & & & \\ 0 & & & & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix}.$$

First suppose that $a_{22} = 0$. Then Lemma 3.1 implies that $a_{23} = a_{32} = 0$. Since each row of A has three 1's, there exist $l > j > 3$ such that $a_{2j} = a_{2l} = 1$. Since each column of A has three 1's, there exist $k > i > 3$ such that $a_{i2} = a_{k2} = 1$, and there exist $q > p > 3$ such that $a_{p3} = a_{q3} = 1$. Since column j contains only three 1's, we must have, by Lemma 3.1, that $p = i$ and $q = k$. But then row i has at least four 1's, a contradiction. Therefore we have $a_{22} = 1$.

We now focus on a_{23}.

Case I: Assume that $a_{23} = 0$. Let the third 1 in row 2 occur in column $j \geq 3$. There exist integers $k > i \geq 3$ such that $a_{k3} = a_{i3} = 1$. Since A is minimal, we must have $a_{kj} = a_{ij} = 1$. If $j > 4$, then $a_{24} = 0$, contradicting Lemma 3.1. Hence $j = 4$. Using Lemma 3.1 and a little thought, we see that $k = i + 1$ and $i \in \{3, 4\}$. Thus the

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submatrix $A[\{i, i+1\}, \{3, 4\}]$ at the intersection of rows i and $i+1$ and columns 3 and 4 equals J_2, and this submatrix intersects row 3 or row 4. We now consider two subcases according to the value of a_{32}.

First suppose that $a_{32} = 1$. Since row 3 has only three 1's, we see that $i = 4$, and applying Lemma 3.1 we see that $a_{35} = 1$. Thus A has the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 1 & & \cdots & \\ 0 & 0 & 1 & 1 & & \cdots & \\ 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & 0 & & & \end{bmatrix}.$$

Applying Lemma 3.1 to A^T, we see that $a_{45} = a_{55} = 1$. Hence $A = V^T \oplus A'$ for some A'.

Now suppose that $a_{32} = 0$. Recall that $i \in \{3, 4\}$. Suppose that $i = 4$. Since each column contains only three 1's, we have $a_{33} = a_{34} = 0$. Applying Lemma 3.1, we get that $a_{35} = a_{36} = 1$. Since A cannot have a submatrix equal to L_2, we conclude that $a_{45} = a_{55} = 1$, giving four 1's in row 4. Therefore we must have $i = 3$. Now A has the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & & \\ 1 & 1 & 0 & 1 & & \\ 1 & 0 & 1 & 1 & & \\ 0 & a_{42} & 1 & 1 & & \\ & & & & & \\ & & & & & \end{bmatrix}.$$

Since $a_{12} + a_{22} + a_{32} = a_{13} + a_{23} + a_{33}$ and $a_{43} = 1$, we have, from Lemma 3.1, that $a_{42} = 1$, and $A = F_4 \oplus A'$ for some A'.

Case II: Assume that $a_{23} = 1$.

First suppose that $a_{32} = 0$, and so by Lemma 3.1, $a_{33} = 0$. Since rows 1 and 2 contain only 0's beyond column 3, and since row 4 contains three 1's, it again follows from Lemma 3.1 that $a_{34} = a_{35} = 1$. Since $a_{i1} = 0$ for all $i \geq 4$, applying Lemma 3.1 to A^T, we have $a_{42} = 1$, and to avoid L_2, we also have $a_{44} = a_{45} = 1$, and so $a_{43} = 0$. Since $a_{i1} = a_{i2} = 0$ for all $i \geq 5$, we have $a_{53} = 1$ by Lemma 3.1 applied to A^T, and using Lemma 3.1 again we see that $a_{54} = a_{55} = 1$. Therefore, $A = V \oplus A'$ for some matrix A'.

We now suppose that $a_{32} = 1$ so that A begins with the form

$$\begin{bmatrix} 1 & 1 & 1 & & \\ 1 & 1 & 1 & & \\ 1 & 1 & a_{33} & & \\ & & & & \\ & & & & \end{bmatrix}.$$

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If $a_{33} = 1$, then $A = J_3 \oplus A'$ for some matrix A'. Now assume that $a_{33} = 0$. By Lemma 3.1 we must have $a_{34} = 1$ and, by considering A^T, $a_{43} = 1$. Hence also $a_{44} = 1$. It follows also from Lemma 3.1, using the fact that row 4 contains a 1 in some column k with $k \geq 5$, that $a_{45} = a_{54} = 1$. Suppose that $a_{55} = 0$. Then $a_{56} = a_{57} = 1$ by Lemma 3.1, and by symmetry, $a_{65} = a_{75} = 1$, implying also that $a_{66} = a_{67} = a_{76} = a_{77} = 1$. Hence A has the form

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}.$$

Therefore, $A = U_1 \oplus A'$ for some matrix A'. Now suppose that $a_{55} = 1$. Then $a_{56} = a_{65} = 1$. If $a_{66} = 0$, then arguing as above we see that $A = U_2 \oplus A'$ for some matrix A'. Otherwise we continue and eventually see that $A = U_i \oplus A'$ for some integer i and matrix A'. $\square$

It would be interesting to characterize all minimal matrices in the Bruhat order for $k \geq 4$ as done for $k = 2$ and $k = 3$. To do this would require a characterization, for all $k \leq n$, of all minimal matrices in $\mathcal{A}(n,k)$ which cannot be expressed as a nontrivial direct sum. But even for $k = 4$, this appears difficult. For example, the following matrices are minimal matrices in $\mathcal{A}(n,4)$ for an appropriate n that cannot be expressed as a nontrivial direct sum.

$$\begin{bmatrix} 1&1&1&1&&&&&&&&&&&& \\ 1&1&1&1&&&&&&&&&&&& \\ 1&1&1&1&&&&&&&&&&&& \\ 1&1&1&0&1&&&&&&&&&&& \\ &&&1&1&1&1&0&&&&&&&& \\ &&&&1&1&1&1&&&&&&&& \\ &&&&1&1&1&1&&&&&&&& \\ &&&&0&1&1&1&1&&&&&&& \\ &&&&&&&1&1&1&1&0&&&& \\ &&&&&&&&1&1&1&1&&&& \\ &&&&&&&&1&1&1&1&&&& \\ &&&&&&&&0&1&1&1&1&&& \\ &&&&&&&&&&&1&0&1&1&1 \\ &&&&&&&&&&&&1&1&1&1 \\ &&&&&&&&&&&&1&1&1&1 \\ &&&&&&&&&&&&1&1&1&1 \end{bmatrix},$$

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ELA
http://math.technion.ac.il/iic/ela

$$\left[\begin{array}{cccccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & & & \\
1 & 1 & 1 & 1 & & & & & & & & & & \\
1 & 1 & 1 & 0 & 1 & & & & & & & & & \\
1 & 1 & 0 & 0 & 1 & 1 & & & & & & & & \\
 & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & 1 & 1 & 1 & 1 & & & & & & & \\
 & & & & & 1 & 1 & 1 & 1 & 0 & & & & \\
 & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & 0 & 1 & 1 & 1 & 1 & & & \\
 & & & & & & & & & 1 & 0 & 1 & 1 & 1 \\
 & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & 1 & 1 & 1 & 1
\end{array}\right],$$

and

$$\left[\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & & & & & & & \\
1 & 1 & 1 & 1 & & & & & & & \\
1 & 1 & 1 & 0 & 1 & & & & & & \\
1 & 1 & 0 & 0 & 1 & 1 & & & & & \\
 & & 1 & 1 & 1 & 1 & 0 & & & & \\
 & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & 0 & 1 & 1 & 1 & 1 & & \\
 & & & & & & 1 & 1 & 0 & 1 & 1 \\
 & & & & & & 1 & 0 & 1 & 1 & 1 \\
 & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & 1 & 1 & 1 & 1
\end{array}\right].$$

In fact, there are many more that can be constructed.

We conclude this note with a conjecture. By Corollary 1.1, a minimal matrix in $\mathcal{A}(R,S)$ has no submatrix equal to L_2. We conjecture that the converse holds.

CONJECTURE. A matrix in $\mathcal{A}(R,S)$ that does not have L_2 as a submatrix is minimal in the Bruhat order on $\mathcal{A}(R,S)$.

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