

CONTROLLABILITY OF SERIES CONNECTIONS*

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Abstract. In this paper the controllability of series connections of arbitrary many linear systems is studied. As the main result, necessary and sufficient conditions are given, under which the system obtained as a result of series connections of arbitrary many linear systems is controllable.

Key words. Controllability, Linear systems, Completion, Invariant factors.

AMS subject classifications. 93B05, 15A21.

1. Introduction. Let S_i be a time-invariant linear system, with state x_i , input u_i and output y_i , $i = 1, \dots, m$:

$$\xrightarrow{u_i} \boxed{S_i} \xrightarrow{y_i} \quad (1.1)$$

Suppose that the system S_i is described by the following system of linear differential equations:

$$\dot{x}_i = A_i x_i + B_i u_i, \quad (1.2)$$

$$y_i = C_i x_i, \quad (1.3)$$

where $A_i \in \mathbb{K}^{n_i \times n_i}$, $B_i \in \mathbb{K}^{n_i \times m_i}$, $C_i \in \mathbb{K}^{p_i \times n_i}$, $i = 1, \dots, m$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$; for details see e.g., [3].

Let $j \in \{1, \dots, m\}$. The algebraic properties of the system S_j depend on the properties of the triple of matrices (A_j, B_j, C_j) . Recall that the system S_j is *controllable* if and only if the pair (A_j, B_j) is *controllable*, where the controllability of a pair is defined as follows:

DEFINITION 1.1. Let \mathbb{F} be a field. Let $A_j \in \mathbb{F}^{n_j \times n_j}$, $B_j \in \mathbb{F}^{n_j \times m_j}$. The pair (A_j, B_j) is said to be *controllable* if one of the following (equivalent) conditions is satisfied:

- 1) $\min_{\lambda \in \mathbb{F}} \text{rank} \begin{bmatrix} \lambda I - A_j & -B_j \end{bmatrix} = n_j$
 - 2) all invariant factors of the matrix pencil $\begin{bmatrix} \lambda I - A_j & -B_j \end{bmatrix}$ are trivial
 - 3) $\text{rank} \begin{bmatrix} B_j & A_j B_j & A_j^2 B_j & \dots & A_j^{n_j-1} B_j \end{bmatrix} = n_j$.
- (1.4)

In this case, we also say that the matrix $\begin{bmatrix} A_j & B_j \end{bmatrix}$ and the corresponding matrix pencil $\begin{bmatrix} \lambda I - A_j & -B_j \end{bmatrix}$ are controllable.

*Received by the editors 6 March 2007. Accepted for publication 13 June 2007. Handling Editor: Joao Filipe Queiro.

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By series connections of the linear systems S_1, \dots, S_m we mean connections where the input of the system S_{i+1} is a linear function of the output of S_i , $i = 1, \dots, m - 1$, i.e.,

$$u_{i+1} = X_i y_i, \quad i = 1, \dots, m - 1, \quad (1.5)$$

where $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$. As a result of this connection, we obtain a new linear system S , with input u_1 , output y_m and state $[x_1^T \ \dots \ x_m^T]^T$.

Thus, studying the properties of the system S , arise the following matrix completion control problem:

PROBLEM 1.2. *Let \mathbb{F} be a field. Find necessary and sufficient conditions for the existence of matrices $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$, $i = 1, \dots, m - 1$, such that the matrix*

$$\left[\begin{array}{cccc|c} A_1 & 0 & 0 & \cdots & 0 \\ B_2 X_1 C_1 & A_2 & 0 & \ddots & 0 \\ 0 & B_3 X_2 C_2 & A_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & B_m X_{m-1} C_{m-1} & A_m \end{array} \right] \begin{array}{c} B_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{array} \quad (1.6)$$

is controllable.

In Section 4 (Theorem 4.1), we give a complete solution to Problem 1.2 when \mathbb{F} is an infinite field. Furthermore, in Section 5, we obtain solutions over arbitrary fields of particular cases of the previous problem.

Similar problems, especially in the case $m = 2$, have been studied previously; see for example the results of I. Baragaña and I. Zaballa [1], and F. C. Silva [8].

2. Notation and Auxiliary results. Let \mathbb{F} be a field. For any polynomial $f \in \mathbb{F}[\lambda]$, $d(f)$ denotes its degree. If $f(\lambda) = \lambda^k - a_{k-1}\lambda^{k-1} - \dots - a_1\lambda - a_0 \in \mathbb{F}[\lambda]$, where $k > 0$, then the matrix

$$C(f(\lambda)) := \left[\begin{array}{cccc} e_2^{(k)} & \cdots & e_k^{(k)} & a \end{array} \right]^T,$$

where $e_i^{(k)}$ is the i th column of the identity matrix I_k and $a = [a_0 \ \dots \ a_{k-1}]^T$, is called the companion matrix for the polynomial $f(\lambda)$.

If $A(\lambda) \in \mathbb{F}[\lambda]^{n \times m}$, with $r = \text{rank } A(\lambda)$, and $\psi_1 | \dots | \psi_r$ are the invariant factors of $A(\lambda)$, make a convention that $\psi_i = 1$ for $i \leq 0$ and $\psi_i = 0$ for $i \geq r + 1$.

DEFINITION 2.1. Let $A, A' \in \mathbb{F}^{n \times n}$, $B, B' \in \mathbb{F}^{n \times l}$. Two matrices

$$M = \left[\begin{array}{cc} A & B \end{array} \right], \quad M' = \left[\begin{array}{cc} A' & B' \end{array} \right] \quad (2.1)$$

are feedback equivalent if there exists a nonsingular matrix

$$P = \left[\begin{array}{cc} N & 0 \\ V & T \end{array} \right]$$

where $N \in \mathbb{F}^{n \times n}$, $V \in \mathbb{F}^{l \times n}$, $T \in \mathbb{F}^{l \times l}$, such that $M' = N^{-1}MP$.

If M and M' are feedback equivalent, then we also say that the corresponding pairs (A, B) and (A', B') are feedback equivalent.

It is easy to verify that two matrices M and M' are feedback equivalent if and only if the corresponding matrix pencils

$$R = \begin{bmatrix} \lambda I - A & -B \end{bmatrix} \quad \text{and} \quad R' = \begin{bmatrix} \lambda I - A' & -B' \end{bmatrix} \quad (2.2)$$

are strictly equivalent, for details see [4].

If $(A, B) \in \mathbb{F}^{n \times n} \times \mathbb{F}^{n \times m}$, is a controllable matrix pair, then it is feedback equivalent to the pair (A_c, B_c) with

$$A_c = \text{diag}(A_1, \dots, A_s), \quad B_c = \begin{bmatrix} \text{diag}(B_1, \dots, B_s) & 0 \end{bmatrix},$$

where

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{F}^{k_i \times k_i}, \quad B_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{F}^{k_i \times 1}, \quad 1 \leq i \leq s.$$

The pair (A_c, B_c) is called *the Brunovsky canonical form* of the pair (A, B) , and the positive integers $k_1 \geq \dots \geq k_s$ are called *the nonzero controllability indices* of (A, B) . Analogously as in [2], we introduce the following definition.

DEFINITION 2.2. Two polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{p \times q}$ and $B(\lambda) \in \mathbb{F}[\lambda]^{p \times q}$ are *SP-equivalent* if there exist invertible matrices $P \in \mathbb{F}^{p \times p}$ and $Q(\lambda) \in \mathbb{F}[\lambda]^{q \times q}$ such that

$$PA(\lambda)Q(\lambda) = B(\lambda).$$

LEMMA 2.3. [2] *Let \mathbb{F} be an infinite field and $f(x), g(x), h(x)$ be nonzero polynomials over \mathbb{F} . Then there exists $\alpha \in \mathbb{F}$ such that*

$$\gcd(f(x) + \alpha g(x), h(x)) = \gcd(f(x), g(x), h(x)). \quad (2.3)$$

In fact, in [2] was proved that (2.3) is not valid only for finitely many $\alpha \in \mathbb{F}$. Hence, (2.3) is valid for a generic (almost every) $\alpha \in \mathbb{F}$.

PROPOSITION 2.4. [6, 7, 9] *Let \mathbb{D} be a principal ideal domain. Let $A \in \mathbb{D}^{n \times n}$, $B \in \mathbb{D}^{n \times n}$. Let $\alpha_1 | \dots | \alpha_n$ be the invariant factors of A , and $\beta_1 | \dots | \beta_n$ be the invariant factors of B . Let $\gamma_1 | \dots | \gamma_n$ be the invariant factors of AB . Then we have*

$$\text{lcm}(\alpha_{n-k-i+1}\beta_{i+1} : 0 \leq i \leq n-k) \mid \gamma_{n-k+1} \mid \gcd(\alpha_{n-i+1}\beta_{n-k+i} : 1 \leq i \leq k), \quad (2.4)$$

$$k = 1, \dots, n.$$

3. Preliminary results. The following proposition deals with the almost canonical form for the SP equivalence of arbitrary square polynomial matrix. Proof goes analogously as the proof of Proposition 2 in [2], thus will be omitted.

PROPOSITION 3.1. *Let \mathbb{F} be an infinite field and let $A(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$. Let $r = \text{rank } A(\lambda) \leq n$. Then $A(\lambda)$ is SP-equivalent to a lower triangular matrix $S(\lambda) = (s_{ij}(\lambda))$, $i, j \in \{1, \dots, n\}$, with the following properties:*

1. $s_{ii}(\lambda) = s_i(\lambda)$, $i = 1, \dots, r - 1$, where $s_1(\lambda) | \dots | s_{r-1}(\lambda)$ are the first $r - 1$ invariant factors of $A(\lambda)$
2. $s_{ii}(\lambda) | s_{ji}(\lambda)$, $1 \leq i \leq r - 1$, $i \leq j \leq n$
3. $s_r(\lambda) = \text{gcd}(s_{rr}(\lambda), \dots, s_{nr}(\lambda))$ and $d(s_{rr}(\lambda)) \geq \dots \geq d(s_{nr}(\lambda))$ where $s_r(\lambda)$ is the r -th invariant factor of $A(\lambda)$
4. if $i \leq r - 1$ and $i < j$ and $s_{ji}(\lambda) \neq 0$, then $s_{ji}(\lambda)$ is monic and $d(s_{ii}(\lambda)) < d(s_{ji}(\lambda))$
5. $s_{ij} = 0$, $j > r$.

Further on, the matrix $S(\lambda)$ will be called *the SP canonical form* of the matrix $A(\lambda)$.

LEMMA 3.2. *Let \mathbb{F} be a field. Let $A \in \mathbb{F}^{n \times n}$, $B \in \mathbb{F}^{n \times m}$. If there exists $X \in \mathbb{F}^{m \times p}$ such that (A, BX) is controllable, then the pair (A, B) is controllable.*

Proof. There exists an invertible matrix $P \in \mathbb{F}^{n \times n}$ such that

$$PBX = \begin{bmatrix} Y \\ 0 \end{bmatrix}, \quad Y \in \mathbb{F}^{\text{rank } B \times p}.$$

Thus, from the controllability of (A, BX) and since

$$\begin{bmatrix} PAP^{-1} & PBX \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & Y \\ A_3 & A_4 & 0 \end{bmatrix}, \quad A_4 \in \mathbb{F}^{(n-\text{rank } B) \times (n-\text{rank } B)},$$

we have that the pair (A_4, A_3) is controllable. Furthermore, there exists an invertible matrix $Q \in \mathbb{F}^{m \times m}$ such that

$$PBQ = \begin{bmatrix} I_{\text{rank } B} & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, the controllability of (A, B) is equivalent to the controllability of (A_4, A_3) , which concludes our proof. \square

LEMMA 3.3. *Let \mathbb{F} be an infinite field. Let $A(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{a \times b}$ be such that $n = \text{rank } A(\lambda)$. Let $\alpha_1 | \dots | \alpha_n$ be the invariant factors of $A(\lambda)$, and let $\beta_1 | \dots | \beta_s$ be the invariant factors of $C(\lambda)$, where $s = \text{rank } C(\lambda)$.*

There exists $X \in \mathbb{F}^{n \times a}$ such that

$$\begin{bmatrix} A(\lambda) & XC(\lambda) \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad (3.1)$$

if and only if

$$\text{gcd}(\alpha_i, \beta_{n+1-i}) = 1, \quad i = 1, \dots, n. \quad (3.2)$$

Proof. If $a \leq b$, then there exists an invertible matrix $Q(\lambda) \in \mathbb{F}[\lambda]^{b \times b}$, such that

$$C(\lambda)Q(\lambda) = \begin{bmatrix} D(\lambda) & 0 \end{bmatrix}, \text{ where } D(\lambda) \in \mathbb{F}[\lambda]^{a \times a}.$$

Thus, instead of $C(\lambda)$ we can consider the matrix $D(\lambda)$.

If $a > b$, then instead of the matrix $C(\lambda)$ consider the matrix

$$D(\lambda) = \begin{bmatrix} C(\lambda) & 0 \end{bmatrix}, \text{ where } D(\lambda) \in \mathbb{F}[\lambda]^{a \times a}.$$

Thus, without loss of generality, we can assume that $a = b$.

Necessity:

Suppose that there exists $X \in \mathbb{F}^{n \times a}$, such that $\begin{bmatrix} A(\lambda) & XC(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$. Denote by $A'(\lambda)$ and $C'(\lambda)$ the Smith canonical forms of the matrices $A(\lambda)$ and $C(\lambda)$, respectively. Then $\begin{bmatrix} A(\lambda) & XC(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} A'(\lambda) & X(\lambda)C'(\lambda) \end{bmatrix}$, for some $X(\lambda) \in \mathbb{F}[\lambda]^{n \times a}$. Thus, we have that for every $x \in \bar{\mathbb{F}}$, $\text{rank} \begin{bmatrix} A'(x) & X(x)C'(x) \end{bmatrix} = n$.

If $\text{gcd}(\alpha_n, \beta_s) = 1$, then the condition is obviously satisfied. Otherwise, let $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, s\}$ be such that $\text{gcd}(\alpha_i, \beta_j) \neq 1$. Let $\lambda_0 \in \bar{\mathbb{F}}$ be a common zero of α_i and β_j . Let $t := \min_{k \in \{1, \dots, n\}} \{k | \alpha_k(\lambda_0) = 0\}$ and $p := \min_{l \in \{1, \dots, s\}} \{l | \beta_l(\lambda_0) = 0\}$. The rank of the matrix $\begin{bmatrix} A'(\lambda_0) & X(\lambda_0)C'(\lambda_0) \end{bmatrix}$ (which is equal to n) is less or equal than the number of its nonzero columns. Since the number of nonzero columns of $A'(\lambda_0)$ is $t-1$ and the number of nonzero columns of $C'(\lambda_0)$ is $p-1$, we have

$$n \leq t-1 + p-1, \text{ and so } i+j \geq n+2.$$

Thus, for all indices i and j such that $i+j \leq n+1$, the polynomials α_i and β_j are mutually prime, which proves our condition.

Sufficiency:

Suppose that the condition (3.2) is satisfied. Without loss of generality, we shall consider $A(\lambda)$ in its SP canonical form, and $C(\lambda)$ in its SP equivalent form $M(\lambda)$ which we describe below:

First, put the matrix $C(\lambda) \in \mathbb{F}[\lambda]^{a \times a}$, into its SP canonical form:

$$\left[\begin{array}{cccc|c} \beta_1 & 0 & & 0 & 0 \\ a_{11}\beta_1 & \beta_2 & & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ a_{s-21}\beta_1 & a_{s-22}\beta_2 & \cdots & \beta_{s-1} & 0 \\ a_{s-11}\beta_1 & a_{s-12}\beta_2 & \cdots & a_{s-1s-1}\beta_{s-1} & X_0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{a-11}\beta_1 & a_{a-12}\beta_2 & \cdots & a_{a-1s-1}\beta_{s-1} & X_{a-s} \end{array} \right], \quad (3.3)$$

where $a_{ij} \in \mathbb{F}[\lambda]$, $1 \leq i \leq a-1$, $1 \leq j \leq s-1$, and $\text{gcd}(X_0, \dots, X_{a-s}) = \beta_s$. By using the condition and Lemma 2.3, there exist $x_1, \dots, x_{a-s} \in \mathbb{F}$, such that

$$\text{gcd}(\alpha_{n-s+1}, X_0 + x_1X_1 + \cdots + x_{a-s}X_{a-s}) = 1.$$

Let $\bar{\beta}_s := X_0 + x_1 X_1 + \cdots + x_{a-s} X_{a-s}$. By multiplying the row $s + i$ by x_i , for all $i = 1, \dots, a - s$, and adding it to the s th row, we obtain the matrix $M(\lambda)$, which is SP equivalent to the matrix $C(\lambda)$, and at the position (s, s) has the polynomial $\bar{\beta}_s$. Further on, the matrix $M(\lambda)$ will be called *the SP-quasi canonical form* of the matrix $C(\lambda)$. Note that $\beta_s | \bar{\beta}_s$ and $\gcd(\alpha_{n-s+1}, \bar{\beta}_s) = 1$.

Consider the submatrix $\bar{M}(\lambda)$ of $M(\lambda)$ formed by the rows $2, \dots, a - 1$, and by the columns $2, \dots, a - 1$. If $s = a$, the invariant factors of $\bar{M}(\lambda)$ are $\beta_2 | \cdots | \beta_{s-1}$ and if $s < a$, the invariant factors of $\bar{M}(\lambda)$ are $\beta_2 | \cdots | \beta_{s-1} | \beta'_s$, for some polynomial β'_s which satisfies $\beta_s | \beta'_s | \bar{\beta}_s$.

From now on, we shall consider the matrix $M(\lambda)$ instead of the matrix $C(\lambda)$ in (3.1). The proof is further split into three cases:

Case 1. Let $n = a$.

The proof goes by induction on n . The case $n = 1$ is trivial. If $n = 2$, there are two nontrivial possibilities on s : $s = 1$ or $s = 2$.

If $s = 2$, it is enough to prove the existence of $x \in \mathbb{F}$, such that the matrix

$$\left[\begin{array}{cc|cc} \alpha_1 & 0 & \beta_1 & 0 \\ b(\lambda)\alpha_1 & \alpha_2 & (a(\lambda) + x)\beta_1 & \beta_2 \end{array} \right] \quad (3.4)$$

has two invariant factors both equal to 1, where $a(\lambda), b(\lambda) \in \mathbb{F}[\lambda]$.

In fact, we shall prove that there exists $x \in \mathbb{F}$ such that the second determinantal divisor of (3.4), D_2 , given by

$$D_2 = \gcd(\beta_1\beta_2, \alpha_1\alpha_2, \beta_1\alpha_2, \alpha_1\beta_2, \alpha_1\beta_1(b(\lambda) - a(\lambda) - x)),$$

is equal to 1.

Since \mathbb{F} is infinite, by applying Lemma 2.3, there exists $x \in \mathbb{F}$, such that

$$D_2 = \gcd(\beta_1\beta_2, \alpha_1\alpha_2, \beta_1\alpha_2, \alpha_1\beta_2, \alpha_1\beta_1).$$

Since $\gcd(\beta_1, \alpha_2) = 1$ and $\gcd(\beta_2, \alpha_1) = 1$, we have $D_2 = 1$, as wanted.

If $s = 1$, we need to prove the existence of $x \in \mathbb{F}$ such that the second determinantal divisor of the matrix

$$\left[\begin{array}{cc|cc} 1 & 0 & p(\lambda) & 0 \\ b(\lambda) & \alpha_2 & q(\lambda) + xp(\lambda) & 0 \end{array} \right]$$

is equal to 1, whenever $\gcd(p(\lambda), \alpha_2) = 1$, $b(\lambda), p(\lambda), q(\lambda) \in \mathbb{F}[\lambda]$. By simple calculation, we have $D_2 = \gcd(p(\lambda)b(\lambda) - q(\lambda) - xp(\lambda), \alpha_2, p(\lambda)\alpha_2, \alpha_2)$. Thus, again by applying Lemma 2.3, we obtain the existence of $x \in \mathbb{F}$ such that $D_2 = 1$.

Now suppose that the claim is true for $n - 2$ and prove that it will be valid for n .

Let $\bar{A}(\lambda)$ be a submatrix of $A(\lambda)$ formed by the rows $2, \dots, n - 1$ and the columns $2, \dots, n - 1$. Thus, $\bar{A}(\lambda)$ has $\alpha_2 | \cdots | \alpha_{n-1}$ as the invariant factors. In both cases, $s = a$ or $s < a$, the invariant factors of $\bar{M}(\lambda)$ and of $\bar{A}(\lambda)$ satisfy the condition (3.2). Thus, we can apply the induction hypothesis and obtain that there exists $Y \in \mathbb{F}^{(n-2) \times (n-2)}$ such that the matrix $\begin{bmatrix} \bar{A}(\lambda) & Y\bar{M}(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_{n-2} & 0 \end{bmatrix}$.

To finish the proof, we shall show that there exists $x \in \mathbb{F}$, such that the matrix $\begin{bmatrix} A(\lambda) & XM(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$, where

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & Y & 0 \\ x & 0 & 1 \end{bmatrix}.$$

Since $\begin{bmatrix} \bar{A}(\lambda) & Y\bar{M}(\lambda) \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_{n-2} & 0 \end{bmatrix}$, and from the forms of matrices $A(\lambda)$ and $M(\lambda)$, the matrix $\begin{bmatrix} A(\lambda) & XM(\lambda) \end{bmatrix}$ is equivalent to the following one

$$\left[\begin{array}{c|c|c|c|c|c} \alpha_1 & 0 & 0 & \beta_1 & 0 & 0 \\ \hline 0 & I_{n-2} & 0 & 0 & 0 & 0 \\ \hline p(\lambda)\alpha_1 & 0 & \alpha_n & (q(\lambda) + x)\beta_1 & * & \beta_n \end{array} \right], \quad (3.5)$$

for some polynomials $p(\lambda)$ and $q(\lambda) \in \mathbb{F}[\lambda]$ (* denotes unimportant entries).

The matrices

$$\begin{bmatrix} \alpha_1 & 0 \\ p(\lambda)\alpha_1 & \alpha_n \end{bmatrix} \text{ and } \begin{bmatrix} \beta_1 & 0 \\ q(\lambda)\beta_1 & \beta_n \end{bmatrix}$$

have $\alpha_1|\alpha_n$ and $\beta_1|\beta_n$ as the invariant factors, respectively, and they are both in SP canonical forms.

Since $\gcd(\alpha_1, \beta_n) = \gcd(\alpha_n, \beta_1) = 1$ by applying the case $n = 2$, there exists $x \in \mathbb{F}$ such that

$$\left[\begin{array}{c|c} \alpha_1 & 0 \\ \hline p(\lambda)\alpha_1 & \alpha_n \end{array} \middle| \begin{array}{c|c} \beta_1 & 0 \\ \hline (q(\lambda) + x)\beta_1 & \beta_n \end{array} \right]$$

is equivalent to $\begin{bmatrix} I_2 & 0 \end{bmatrix}$.

Hence, for such $x \in \mathbb{F}$ we have that the matrix (3.5) is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$, as wanted.

Case 2. Let $n > a$.

Let

$$\tilde{M}(\lambda) = \begin{bmatrix} I_a \\ 0 \end{bmatrix} M(\lambda) \begin{bmatrix} I_a & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{n \times n}.$$

Then the invariant factors of $\tilde{M}(\lambda)$ are $\beta_1|\dots|\beta_s$. From the Case 1., there exists $Y \in \mathbb{F}^{n \times n}$ such that

$$\begin{bmatrix} A(\lambda) & Y\tilde{M}(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_n & 0 \end{bmatrix}$. Now, put $X := Y \begin{bmatrix} I_a \\ 0 \end{bmatrix} \in \mathbb{F}^{n \times a}$.

Case 3. Let $n < a$.

Let

$$M'(\lambda) = \begin{bmatrix} I_n & 0 \end{bmatrix} M(\lambda) \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{n \times n}.$$

If $n < s$, then the invariant factors of $M'(\lambda)$ are $\beta_1 | \cdots | \beta_n$, and if $n \geq s$, then the invariant factors of $M'(\lambda)$ are $\beta_1 | \cdots | \beta_{s-1} | \beta_s''$, for some polynomial β_s'' such that $\beta_s | \beta_s'' | \beta_s$. By applying the Case 1, there exists $Y \in \mathbb{F}^{n \times n}$ such that

$$\left[\begin{array}{c|c} A(\lambda) & YM'(\lambda) \end{array} \right]$$

is equivalent to $\left[\begin{array}{c|c} I_n & 0 \end{array} \right]$. Now, put $X := Y \left[\begin{array}{c|c} I_n & 0 \end{array} \right] \in \mathbb{F}^{n \times a}$. \square

REMARK 3.4. Let $A(\lambda)$ be in its SP canonical form and $M(\lambda)$ be the SP-quasi canonical form of the matrix $C(\lambda)$. Let $X_0 \in \mathbb{F}^{n \times a}$ be the matrix defined in the previous lemma, such that

$$\left[\begin{array}{c|c} A(\lambda) & X_0 M(\lambda) \end{array} \right] \text{ is equivalent to } \left[\begin{array}{c|c} I_n & 0 \end{array} \right]. \quad (3.6)$$

Let $P \in \mathbb{F}^{n \times n}$ be a lower triangular matrix with units on diagonal. From the proof of Lemma 3.3 (see (3.4)), we have that for a generic matrix P , PX_0 also satisfies (3.6).

Further on in this paper, by S we denote the set of all lower triangular matrices with units on diagonal, P , such that PX_0 satisfies (3.6), and we define

$$G := \{PX_0 | P \in S\}.$$

LEMMA 3.5. Let \mathbb{F} be an infinite field. Let $A(\lambda) \in \mathbb{F}[\lambda]^{n \times n}$ be such that $n = \text{rank } A(\lambda)$, and let $\alpha_1 | \cdots | \alpha_n$ be its invariant factors. Let $D(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$ be such that $m = \text{rank } D(\lambda)$ and let $\beta_1 | \cdots | \beta_m$ be its invariant factors. Let $C(\lambda) \in \mathbb{F}[\lambda]^{a \times n}$, $a \leq m$, and let $\gamma_1 | \cdots | \gamma_s$ be its invariant factors, $s = \text{rank } C(\lambda)$. Let $\mu_1 | \cdots | \mu_n$ be the invariant factors of

$$\left[\begin{array}{c} A(\lambda) \\ C(\lambda) \end{array} \right].$$

If

$$\gcd(\gamma_i, \beta_{m+1-i}) = 1, \quad i = 1, \dots, m, \quad (3.7)$$

then there exists $X \in \mathbb{F}^{m \times a}$, such that

$$\left[\begin{array}{c|c} D(\lambda) & XC(\lambda) \end{array} \right]$$

is equivalent to $\left[\begin{array}{c|c} I_m & 0 \end{array} \right]$, and such that every zero of a polynomial ϵ_{m+i}^X , $i = 1, \dots, n$, is a zero of the polynomial α_i or of the polynomial $\gcd(\beta_j, \mu_{m+i-j+1})$, for some $j = 1, \dots, m$, where $\epsilon_1^X | \cdots | \epsilon_{m+n}^X$ are the invariant factors of

$$T(\lambda) = \left[\begin{array}{c|c} A(\lambda) & 0 \\ \hline XC(\lambda) & D(\lambda) \end{array} \right]. \quad (3.8)$$

Proof. Without loss of generality, consider the matrix $D(\lambda)$ in its SP canonical form, and the matrix $C(\lambda)$ in its SP-quasi canonical form. By the condition (3.7),

and by applying Lemma 3.3 there exists $X_0 \in \mathbb{F}^{m \times a}$, such that $\begin{bmatrix} D(\lambda) & X_0 C(\lambda) \\ I_m & 0 \end{bmatrix}$ is equivalent to $\begin{bmatrix} I_m & 0 \end{bmatrix}$. Even more, by Remark 3.4, for every $X \in G$, the matrix

$$\begin{bmatrix} D(\lambda) & X C(\lambda) \end{bmatrix} \text{ is equivalent to } \begin{bmatrix} I_m & 0 \end{bmatrix}. \quad (3.9)$$

Also, note that for every $X \in G$, the invariant factors of

$$\begin{bmatrix} A(\lambda) \\ X C(\lambda) \end{bmatrix}$$

are exactly $\mu_1 | \cdots | \mu_n$. Indeed, the invariant factors of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+a) \times n}$ are

the same as the invariant factors of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m) \times n}$, since $a \leq m$.

If $a = m$, then by the proof of the previous Lemma, every matrix $X \in G$ is invertible, and so the invariant factors of $\begin{bmatrix} A(\lambda) \\ C(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A(\lambda) \\ X C(\lambda) \end{bmatrix}$ coincide.

If $a < m$, then (see case 2. in the previous lemma) we defined $X := Y \begin{bmatrix} I_a \\ 0 \end{bmatrix}$, where Y is an invertible matrix. Thus,

$$\begin{bmatrix} A(\lambda) \\ X C(\lambda) \end{bmatrix} = \begin{bmatrix} A(\lambda) \\ Y \begin{bmatrix} I_a \\ 0 \end{bmatrix} C(\lambda) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} A(\lambda) \\ C(\lambda) \\ 0 \end{bmatrix},$$

and so its invariant factors are $\mu_1 | \cdots | \mu_n$.

Now, from (3.9), there exist invertible matrices $P(\lambda) \in \mathbb{F}[\lambda]^{m \times m}$, and

$$Q(\lambda) = \begin{bmatrix} Q_1(\lambda) & Q_2(\lambda) \\ Q_3(\lambda) & Q_4(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m) \times (n+m)}, \text{ where } Q_1(\lambda) \in \mathbb{F}[\lambda]^{n \times n},$$

such that

$$\begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix} \left[\begin{array}{c|c} A(\lambda) & 0 \\ \hline X C(\lambda) & D(\lambda) \end{array} \right] Q(\lambda) = \begin{bmatrix} A(\lambda) Q_1(\lambda) & A(\lambda) Q_2(\lambda) \\ 0 & I \end{bmatrix}.$$

Thus, the invariant factors of $A(\lambda) Q_1(\lambda)$ are exactly $\epsilon_{m+1}^X | \cdots | \epsilon_{m+n}^X$. On the other hand, we have that

$$\begin{bmatrix} I & 0 \\ 0 & P(\lambda) \end{bmatrix} \left[\begin{array}{c|c} I & 0 \\ \hline X C(\lambda) & D(\lambda) \end{array} \right] Q(\lambda) = \begin{bmatrix} Q_1(\lambda) & Q_2(\lambda) \\ 0 & I \end{bmatrix}.$$

Hence, the nontrivial invariant factors of $Q_1(\lambda)$ coincide with the nontrivial invariant factors of $D(\lambda)$. So, the invariant factors of the matrix $Q_1(\lambda)$, denoted by $\beta'_1 | \cdots | \beta'_n$, satisfy $\beta'_i = \beta_{i+m-n}$, $i = 1, \dots, n$.

Now, by applying Proposition 2.4 to the matrix product $A(\lambda) Q_1(\lambda)$, we have that for every $X \in G$

$$\epsilon_{i+m}^X | \gcd(\alpha_i \beta_m, \dots, \alpha_n \beta_{i+m-n}), \quad i = 1, \dots, n. \quad (3.10)$$

Denote by $\phi_i := \gcd(\alpha_i\beta_m, \dots, \alpha_n\beta_{i+m-n})$, $i = 1, \dots, n$. Let $\lambda_1^i, \dots, \lambda_{k_i}^i \in \bar{\mathbb{F}}$ be distinct zeros of ϕ_i , $i = 1, \dots, n$.

Let $l \in \{1, \dots, k_i\}$. We shall show the following:

(*) If λ_l^i is not a zero of the polynomial $\alpha_i \prod_{j=1}^m \gcd(\beta_j, \mu_{m+i-j+1})$, then for a generic $X \in G$, λ_l^i is not a zero of the corresponding ϵ_{m+i}^X .

This will obviously prove that for generic $X \in G$, for every $i = 1, \dots, n$, every zero of the polynomial ϵ_{m+i}^X is a zero of the polynomial α_i or of the polynomial $\gcd(\beta_j, \mu_{m+i-j+1})$ for some $j = 1, \dots, m$, as wanted.

Thus, we are left with proving (*).

Let $i \in \{1, \dots, n\}$, $l \in \{1, \dots, k_i\}$, and λ_l^i be a zero of ϕ_i such that $\alpha_i(\lambda_l^i) \neq 0$ and $\gcd(\beta_j, \mu_{m+i-j+1})(\lambda_l^i) \neq 0$, for all $j = 1, \dots, m$. Let

$$p = \min_{w=i, \dots, n+1} \{w | \alpha_w(\lambda_l^i) = 0\}$$

$$t = \min_{w=i+m-n, \dots, m+1} \{w | \beta_w(\lambda_l^i) = 0\}.$$

Since $\phi_i(\lambda_l^i) = 0$, we have $\alpha_{p-1}(\lambda_l^i) \neq 0 \Rightarrow \beta_{i+m-p+1}(\lambda_l^i) = 0$, and $\beta_{t-1}(\lambda_l^i) \neq 0 \Rightarrow \alpha_{i+m-t+1}(\lambda_l^i) = 0$, which gives $p + t \leq i + m + 1$.

Furthermore, since $\alpha_i(\lambda_l^i) \neq 0$ we have $p > i$ and since $\gcd(\beta_{i+m-n}, \mu_{n+1})(\lambda_l^i) = \beta_{i+m-n}(\lambda_l^i) \neq 0$, we have $t > i + m - n$. Also, since $\beta_t(\lambda_l^i) = 0$, we must have $\mu_{m+i-t+1}(\lambda_l^i) \neq 0$.

Consider the following equivalent form of the matrix $T(\lambda_l^i) \in \mathbb{F}^{(n+m) \times (n+m)}$:

$$\left[\begin{array}{cc|ccc} \text{diag}(\alpha_1(\lambda_l^i), \dots, \alpha_{p-1}(\lambda_l^i)) & 0 & & & \\ & 0 & & 0 & \\ \hline & & X\bar{C}(\lambda_l^i) & & \\ & & & \text{diag}(\beta_1(\lambda_l^i), \dots, \beta_{t-1}(\lambda_l^i)) & 0 \\ & & & & 0 \end{array} \right], \quad (3.11)$$

where $\bar{C}(\lambda_l^i) \in \mathbb{F}^{\alpha \times n}$.

Since $X = PX_0$, $P \in S$ (see Remark 3.4), the matrix (3.11) becomes

$$\left[\begin{array}{cc|ccc} \text{diag}(\alpha_1(\lambda_l^i), \dots, \alpha_{p-1}(\lambda_l^i)) & 0 & & & \\ & 0 & & 0 & \\ \hline P \begin{bmatrix} A & B \\ C & D \end{bmatrix} & & & \text{diag}(\beta_1(\lambda_l^i), \dots, \beta_{t-1}(\lambda_l^i)) & 0 \\ & & & & 0 \end{array} \right], \quad (3.12)$$

where

$$X_0\bar{C}(\lambda_l^i) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{F}^{m \times n}, \quad A \in \mathbb{F}^{(t-1) \times (p-1)}.$$

Let

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} := P \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad D' \in \mathbb{F}^{(m-t+1) \times (n-p+1)}.$$

Since, $\mu_{m+i-t+1}(\lambda_l^i) \neq 0$, we have

$$\text{rank} \begin{bmatrix} B \\ D \end{bmatrix} \geq \text{rank} \begin{bmatrix} B' \\ D' \end{bmatrix} = \text{rank} \begin{bmatrix} A(\lambda_l^i) \\ XC(\lambda_l^i) \end{bmatrix} - p + 1 =$$

$$= \text{rank} \begin{bmatrix} \mu_1(\lambda_i^i) & & & \\ & \ddots & & \\ & & \mu_n(\lambda_i^i) & \\ & & & \end{bmatrix} - p + 1 \geq m + i - t - p + 2 \quad (\geq 1).$$

On the other hand, $\epsilon_{m+i}^X(\lambda_i^i) \neq 0$ is equivalent to

$$\text{rank } D' \geq m + i - t - p + 2. \quad (3.13)$$

Indeed, this is because the rank of the matrix (3.11) is equal to $p + t - 2 + \text{rank } D'$. Since $p \geq i + 1$ and $t \geq i + m - n + 1$, we have

$$\min\{m - t + 1, n - p + 1\} \geq m + i - p - t + 2.$$

Thus, for a generic matrix $X \in G$, we have that (3.13) is valid, which finishes our proof. \square

4. Main result. The following theorem gives our main result:

THEOREM 4.1. *Let \mathbb{F} be an infinite field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $B_i \in \mathbb{F}^{n_i \times m_i}$, $i = 1, \dots, m$, $C_i \in \mathbb{F}^{p_i \times n_i}$, $i = 1, \dots, m - 1$. There exist matrices $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$, $i = 1, \dots, m - 1$, such that*

$$M = \left[\begin{array}{cccc|cc} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ B_2 X_1 C_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3 X_2 C_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & B_m X_{m-1} C_{m-1} & A_m & 0 \end{array} \right] \quad (4.1)$$

is controllable if and only if:

$$(A_i, B_i) \text{ are controllable for all } i = 1, \dots, m, \quad (4.2)$$

and

$$\text{gcd}(\gamma_{k_1}^i, \mu_{k_2}^{i+1}, \dots, \mu_{k_{j-i}}^{j-1}, \alpha_{k_{j-i+1}}^j) = 1, \quad 1 \leq i < j \leq m, \quad (4.3)$$

for all indices k_1, \dots, k_{j-i+1} such that

$$k_1 + \dots + k_{j-i+1} \leq n_i + \dots + n_j + j - i.$$

Here $\gamma_1^i | \dots | \gamma_{y_i}^i$ are the invariant factors of

$$\begin{bmatrix} \lambda I - A_i & -B_i \\ -C_i & 0 \end{bmatrix}, \quad i = 1, \dots, m - 1, \quad (4.4)$$

y_i is its rank, $\alpha_1^i | \dots | \alpha_{n_i}^i$ are the invariant factors of $\lambda I - A_i$, $i = 1, \dots, m$, and $\mu_1^i | \dots | \mu_{n_i}^i$ are the invariant factors of

$$\begin{bmatrix} \lambda I - A_i \\ -C_i \end{bmatrix}, \quad i = 1, \dots, m - 1.$$

Proof.

Necessity:

From the controllability of (4.1), we have that the pair (A_1, B_1) is controllable and $(A_i, B_i X_{i-1} C_{i-1})$ are controllable for all $i = 2, \dots, m$. By applying Lemma 3.2, we obtain the condition (4.2).

Furthermore, there exist invertible matrices $P_i \in \mathbb{F}^{n_i \times n_i}$, such that

$$P_i B_i = \begin{bmatrix} T_i \\ 0 \end{bmatrix}, \quad T_i \in \mathbb{F}^{\text{rank } B_i \times m_i}, \quad i = 1, \dots, m.$$

Let

$$P_i A_i P_i^{-1} = \begin{bmatrix} A_1^i & A_2^i \\ A_3^i & A_4^i \end{bmatrix}, \quad A_1^i \in \mathbb{F}^{\text{rank } B_i \times \text{rank } B_i}, \quad i = 1, \dots, m.$$

Then (A_4^i, A_3^i) is controllable (moreover the controllability of (A_4^i, A_3^i) is equivalent to the controllability of (A_i, B_i)) and there exist invertible matrices $Q_i(\lambda) \in \mathbb{F}[\lambda]^{n_i \times n_i}$ and $S_i(\lambda) \in \mathbb{F}[\lambda]^{n_i \times n_i}$, $i = 1, \dots, m$, such that

$$Q_i(\lambda)(\lambda I - P_i A_i P_i^{-1}) S_i(\lambda) = \begin{bmatrix} A_i(\lambda) & 0 \\ 0 & I \end{bmatrix},$$

where $A_i(\lambda) \in \mathbb{F}[\lambda]^{\text{rank } B_i \times \text{rank } B_i}$, $i = 1, \dots, m$. Note that the first $\text{rank } B_i$ columns of the matrices $Q_i(\lambda)$, $i = 1, \dots, m$ are of the form $\begin{bmatrix} I_{\text{rank } B_i} \\ 0 \end{bmatrix}$. Denote the invariant factors of $A_i(\lambda)$ by $\alpha_1^i | \dots | \alpha_{\text{rank } B_i}^i$, then $\alpha_j^i := \alpha_{j+n_i-\text{rank } B_i}^i$, $j = 1, \dots, \text{rank } B_i$, $i = 1, \dots, m$.

Let

$$Y_i := T_{i+1} X_i \in \mathbb{F}^{\text{rank } B_{i+1} \times p_i}, \quad i = 1, \dots, m-1.$$

Let $\bar{P}_i \in \mathbb{F}^{m_i \times m_i}$ be the invertible matrices such that $-T_i \bar{P}_i = \begin{bmatrix} I_{\text{rank } B_i} & 0 \end{bmatrix}$, $i = 1, \dots, m$. Denote by $P = \text{diag}(P_1, \dots, P_m)$, $Q(\lambda) = \text{diag}(Q_1(\lambda), \dots, Q_m(\lambda))$, $\bar{P} = \text{diag}(P_1^{-1}, \dots, P_m^{-1}, \bar{P}_1)$ and $S(\lambda) = \text{diag}(S_1(\lambda), \dots, S_m(\lambda), I)$.

Furthermore, let $-C_i P_i^{-1} S_i(\lambda) = \begin{bmatrix} C_i(\lambda) & C_i'(\lambda) \end{bmatrix}$, $C_i(\lambda) \in \mathbb{F}[\lambda]^{p_i \times \text{rank } B_i}$, $i = 1, \dots, m-1$.

Now, consider the matrix

$$M(\lambda) = Q(\lambda) P(\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M) \bar{P} S(\lambda).$$

The matrix $M(\lambda)$ has the following form

$$\left[\begin{array}{cc|cc|cc|cc|cc|cc} A_1(\lambda) & 0 & & & & & & & & & I & 0 \\ 0 & I & & & & & & & & & 0 & 0 \\ \hline Y_1 C_1(\lambda) & Y_1 C_1'(\lambda) & A_2(\lambda) & 0 & & & & & & & & \\ 0 & 0 & 0 & I & & & & & & & & \\ \hline & & Y_2 C_2(\lambda) & Y_2 C_2'(\lambda) & A_3(\lambda) & 0 & & & & & & \\ & & 0 & 0 & 0 & I & & & & & & \\ \hline & & & & & \ddots & & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & & \ddots & & & & \\ \hline & & & & & & & & Y_{m-1} C_{m-1}(\lambda) & Y_{m-1} C_{m-1}'(\lambda) & A_m(\lambda) & \\ & & & & & & & & 0 & 0 & 0 & I \end{array} \right].$$

Thus, the matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M$ is equivalent to

$$\left[\begin{array}{cccc|cc|c} Y_1 C_1(\lambda) & A_2(\lambda) & & & & & \\ & Y_2 C_2(\lambda) & A_3(\lambda) & & & & \\ & & Y_3 C_3(\lambda) & A_4(\lambda) & & & \\ & & & \ddots & \ddots & & \\ & & & & Y_{m-2} C_{m-2}(\lambda) & A_{m-1}(\lambda) & \\ \hline & & & & & Y_{m-1} C_{m-1}(\lambda) & A_m(\lambda) \\ \hline & & & & & & I \end{array} \right], \tag{4.5}$$

where nonmarked entries are equal to zero.

Since the matrix

$$\begin{bmatrix} Q_i(\lambda) P_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I - A_i & -B_i \\ -C_i & 0 \end{bmatrix} \begin{bmatrix} P_i^{-1} S_i(\lambda) & 0 \\ 0 & \bar{P}_i \end{bmatrix}$$

is equal to the following one

$$\left[\begin{array}{cc|cc} A_i(\lambda) & 0 & I & 0 \\ 0 & I & 0 & 0 \\ \hline C_i(\lambda) & C_i'(\lambda) & 0 & 0 \end{array} \right], \quad i = 1, \dots, m-1,$$

the invariant factors of the matrix $C_i(\lambda)$, denoted by $\gamma_1^i | \dots | \gamma_{y_i - n_i}^i$, satisfy $\gamma_j^i := \gamma_{j+n_i}^i$, $j = 1, \dots, y_i - n_i$, $i = 1, \dots, m-1$.

Moreover, if denote by $\mu_1^i | \dots | \mu_{\text{rank } B_i}^i$ the invariant factors of

$$\begin{bmatrix} A_i(\lambda) \\ C_i(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(\text{rank } B_i + p_i) \times \text{rank } B_i}, \quad i = 2, \dots, m-1,$$

from

$$\begin{bmatrix} Q_i(\lambda) P_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \lambda I - A_i \\ -C_i \end{bmatrix} S_i(\lambda) = \begin{bmatrix} A_i(\lambda) & 0 \\ 0 & I \\ \hline C_i(\lambda) & C_i'(\lambda) \end{bmatrix},$$

is equivalent to $\begin{bmatrix} I_{\text{rank } B_m} & 0 \\ Y_{m-1}C_{m-1}(\lambda) & A_m(\lambda) \end{bmatrix}$.

Now suppose that the condition is sufficient for $m - 1$ and we shall prove that it is sufficient for m . Consider the matrix

$$\begin{bmatrix} A_{m-1}(\lambda) & 0 \\ Y_{m-1}C_{m-1}(\lambda) & A_m(\lambda) \end{bmatrix}. \quad (4.8)$$

If $p_{m-1} \leq \text{rank } B_m$, by Lemma 3.5, there exists a matrix Y_{m-1} such that the matrix

$$\begin{bmatrix} A_m(\lambda) & Y_{m-1}C_{m-1}(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_{\text{rank } B_m} & 0 \end{bmatrix}$, and every zero of the polynomial $\epsilon_{i+\text{rank } B_m}$ is the zero of the polynomial α_i^{m-1} or of the polynomial $\text{gcd}(\alpha_j^m, \mu_{\text{rank } B_m+i-j+1}^{m-1})$, for some $j = 1, \dots, \text{rank } B_m$, where $\epsilon_1 | \dots | \epsilon_{\text{rank } B_m + \text{rank } B_{m-1}}$ are the invariant factors of (4.8).

If $p_{m-1} > \text{rank } B_m$, instead of $A_m(\lambda)$ consider the matrix $\bar{A}_m(\lambda) := A_m(\lambda) \oplus I_{p_{m-1}-\text{rank } B_m}$. Now, again by Lemma 3.5, there exists a matrix $Y'_{m-1} \in \mathbb{F}^{p_{m-1} \times p_{m-1}}$ such that

$$\begin{bmatrix} \bar{A}_m(\lambda) & Y'_{m-1}C_{m-1}(\lambda) \end{bmatrix}$$

is equivalent to $\begin{bmatrix} I_{p_{m-1}} & 0 \end{bmatrix}$. Then define $Y_{m-1} := \begin{bmatrix} I_{\text{rank } B_m} & 0 \end{bmatrix} Y'_{m-1}$.

In both cases, the matrix (4.8) is equivalent to the matrix

$$\begin{bmatrix} A'_{m-1}(\lambda) & 0 \\ 0 & I_{\text{rank } B_m} \end{bmatrix}, \text{ for some } A'_{m-1}(\lambda) \in \mathbb{F}[\lambda]^{\text{rank } B_{m-1} \times \text{rank } B_{m-1}}.$$

Note that the invariant factors of $A'_{m-1}(\lambda)$, denoted by $\epsilon'_1 | \dots | \epsilon'_{\text{rank } B_{m-1}}$, satisfy $\epsilon'_i = \epsilon_{i+\text{rank } B_m}$, $i = 1, \dots, \text{rank } B_{m-1}$.

Denote the submatrix of (4.5) formed by the rows $1, \dots, \sum_{i=2}^{m-1} \text{rank } B_i$ and by the columns $1, \dots, \sum_{i=1}^{m-2} \text{rank } B_i$, by E . Now, our problem reduces to defining the matrices Y_1, \dots, Y_{m-2} such that the matrix

$$\left[E \mid \begin{array}{c} 0 \\ A'_{m-1}(\lambda) \end{array} \right]$$

is equivalent to $\begin{bmatrix} I & 0 \end{bmatrix}$.

In order to apply the induction hypothesis, and thus to finish the proof, we need to prove the validity of the following condition

$$\text{gcd}(\gamma_{k_1}^i, \mu_{k_2}^{i+1}, \dots, \mu_{k_{m-i-1}}^{m-2}, \epsilon'_{k_{m-i}}) = 1, \quad (4.9)$$

for every $i = 1, \dots, m - 2$ and for all indices k_1, \dots, k_{m-i} such that

$$k_1 + \dots + k_{m-i} \leq \text{rank } B_{i+1} + \dots + \text{rank } B_{m-1} + m - i - 1.$$

Suppose that the condition (4.9) is not valid. Then there exists $\lambda_0 \in \bar{\mathbb{F}}$, a common zero of the polynomials $\gamma_{k_1}^i, \mu_{k_2}^i, \dots, \mu_{k_{m-i-1}}^{m-2}$ and $\epsilon'_{k_{m-i}}$ for some indices k_1, \dots, k_{m-i} satisfying $k_1 + \dots + k_{m-i} \leq \text{rank } B_{i+1} + \dots + \text{rank } B_{m-1} + m - i - 1$.

Hence, λ_0 is a zero of the polynomial $\epsilon_{k_{m-i}+\text{rank } B_m}$. Now, since Y_{m-1} is defined by Lemma 3.5, λ_0 is a zero of the polynomial $\alpha_{k_{m-i}}^{l_{m-1}}$ or of the polynomial $\text{gcd}(\mu_{\text{rank } B_m+k_{m-i}-l+1}^{m-1}, \alpha_l^m)$, for some index $l \in \{1, \dots, \text{rank } B_m\}$.

If $\alpha_{k_{m-i}}^{l_{m-1}}(\lambda_0) = 0$, then

$$\text{gcd}(\gamma_{k_1}^i, \mu_{k_2}^{i+1}, \dots, \mu_{k_{m-i-1}}^{m-2}, \alpha_{k_{m-i}}^{l_{m-1}}) \neq 1$$

which is a contradiction by (4.6).

If $\text{gcd}(\mu_{\text{rank } B_m+k_{m-i}-l+1}^{m-1}, \alpha_l^m)(\lambda_0) = 0$, then

$$\text{gcd}(\gamma_{k_1}^i, \mu_{k_2}^{i+1}, \dots, \mu_{k_{m-i-1}}^{m-2}, \mu_{\text{rank } B_m+k_{m-i}-l+1}^{m-1}, \alpha_l^m) \neq 1,$$

which is again a contradiction. Thus, (4.9) is valid, as wanted. \square

5. Special cases. In this section we study some special cases of the Problem 1.2 over arbitrary fields.

THEOREM 5.1. *Let \mathbb{F} be a field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $B_i \in \mathbb{F}^{n_i \times m_i}$, $i = 1, \dots, m$, $C_i \in \mathbb{F}^{p_i \times n_i}$, $i = 1, \dots, m-1$. Let $\text{rank } B_i = 1$, $i = 1, \dots, m$, and $\text{rank } C_i = 1$, $i = 1, \dots, m-1$. There exist matrices $X_i \in \mathbb{F}^{m_{i+1} \times p_i}$, $i = 1, \dots, m-1$, such that*

$$\left[\begin{array}{cccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ B_2 X_1 C_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3 X_2 C_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & B_m X_{m-1} C_{m-1} & A_m & 0 \end{array} \right] \quad (5.1)$$

is controllable, if and only if

- (i) (A_i, B_i) is controllable, $i = 1, \dots, m$,
- (ii) $\text{gcd}(\gamma_{n_{i+1}}^i, \alpha_{n_j}^j) = 1$, $1 \leq i < j \leq m$,

where $\gamma_1^i | \dots | \gamma_{n_{i+1}}^i$ are the invariant factors of the matrix

$$\begin{bmatrix} \lambda I - A_i & -B_i \\ -C_i & 0 \end{bmatrix}, \quad (5.2)$$

$i = 1, \dots, m-1$. Also, $\alpha_{n_j}^j$ is the only nontrivial invariant factor of $\lambda I - A_j$, $j = 1, \dots, m$.

Proof. First, since $\text{rank } C_i = 1$, $i = 1, \dots, m-1$, there exist invertible matrices $P_i \in \mathbb{F}^{p_i \times p_i}$ such that

$$\bar{C}_i = P_i C_i = \begin{bmatrix} c_i \\ 0 \end{bmatrix}, \text{ where } c_i \in \mathbb{F}^{1 \times n_i}, \quad i = 1, \dots, m-1.$$

Let

$$\bar{X}_i = X_i P_i^{-1}, \quad i = 1, \dots, m - 1.$$

Further on, instead of matrix (5.1), we shall consider the matrix

$$M = \left[\begin{array}{cccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ B_2 \bar{X}_1 \bar{C}_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & B_3 \bar{X}_2 \bar{C}_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & B_m \bar{X}_{m-1} \bar{C}_{m-1} & A_m & 0 \end{array} \right]. \quad (5.3)$$

Necessity:

Suppose that there exist matrices $\bar{X}_1, \dots, \bar{X}_{m-1}$ such that the matrix (5.3) is controllable. Like in Theorem 4.1, we obtain the condition (i). Also, note that the fact that rank of the matrix (5.2) is equal to $n_i + 1$ follows from the controllability of the pair (A_i, B_i) , $i = 1, \dots, m$.

Furthermore, we shall consider the matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M$. As in Theorem 4.1, matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M$ is equivalent to the matrix (4.5), and since in this case $\text{rank } B_i = 1$, $i = 1, \dots, m$, the matrix (4.5) is of the following form

$$\left[\begin{array}{cccccc|c} q_1 & \alpha_{n_2}^2 & & & & & \\ & q_2 & \alpha_{n_3}^3 & & & & \\ & & q_3 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & q_{m-1} & \alpha_{n_m}^m \\ \hline & & & & & & I \end{array} \right], \quad (5.4)$$

where the polynomials $\alpha_{n_i}^i$ are the only nontrivial invariant factors of the matrices $\lambda I - A_i$, $i = 2, \dots, m$, and the polynomials q_i , $i = 1, \dots, m - 1$ (see the proof of Theorem 4.1) are the last nonzero invariant factors of the matrices

$$\begin{bmatrix} \lambda I - A_i & -B_i \\ -\bar{C}_i & 0 \end{bmatrix}, \quad (5.5)$$

i.e.

$$q_i = \gamma_{n_i+1}^i, \quad i = 1, \dots, m - 1.$$

Hence, we have that the matrix

$$N = \left[\begin{array}{cccccc} \gamma_{n_1+1}^1 & \alpha_{n_2}^2 & & & & \\ & \gamma_{n_2+1}^2 & \alpha_{n_3}^3 & & & \\ & & \gamma_{n_3+1}^3 & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & \gamma_{n_{m-1}+1}^{m-1} & \alpha_{n_m}^m \end{array} \right] \quad (5.6)$$

has only trivial invariant factors. Thus, $D_{m-1} = 1$ (D_{m-1} is the $(m-1)$ th determinantal divisor of (5.6), i.e. the greatest common divisor of minors of order $m-1$ of (5.6)). We shall prove that this is equivalent to the condition (ii).

Indeed, since

$$D_{m-1} = \gcd(\Pi_{m-1}, \Pi_{m-2}, \dots, \Pi_0),$$

where

$$\Pi_i := \gamma_{n_1+1}^1 \cdots \gamma_{n_i+1}^i \alpha_{n_{i+2}}^{i+2} \cdots \alpha_{n_m}^m, \quad i = 0, \dots, m-1,$$

we shall prove that:

$$D_{m-1} = 1 \Leftrightarrow \gcd(\gamma_{n_i+1}^i, \alpha_{n_j}^j) = 1, \quad 1 \leq i < j \leq m, \text{ i.e.,}$$

$$D_{m-1} \neq 1 \Leftrightarrow \exists i, j : 1 \leq i < j \leq m, \text{ such that } \gcd(\gamma_{n_i+1}^i, \alpha_{n_j}^j) \neq 1.$$

Let $D_{m-1} \neq 1$. Let $\lambda_0 \in \overline{\mathbb{F}}$ be a common zero of Π_{m-1}, \dots, Π_0 . Since Π_{m-1} is the product only of $\gamma_{n_i+1}^i$'s, at least one of $\gamma_{n_i+1}^i$, $i = 1, \dots, m-1$, must have λ_0 as its zero. Let $k := \min\{i \mid \gamma_{n_i+1}^i(\lambda_0) = 0\}$, $1 \leq k \leq m-1$. Then $\Pi_{k-1} = \gamma_{n_1+1}^1 \cdots \gamma_{n_{k-1}+1}^{k-1} \alpha_{n_{k+1}}^{k+1} \cdots \alpha_{n_m}^m$. Thus, there exists $j > k$ such that $\alpha_{n_j}^j(\lambda_0) = 0$. Then, obviously, $\gcd(\gamma_{n_k+1}^k, \alpha_{n_j}^j) \neq 1$, as wanted.

Conversely, suppose that there exist indices i and j such that $1 \leq i < j \leq m$ and $\gcd(\gamma_{n_i+1}^i, \alpha_{n_j}^j) \neq 1$, i.e., $\exists \lambda_0 \in \overline{\mathbb{F}}$ such that $\gamma_{n_i+1}^i(\lambda_0) = 0$ and $\alpha_{n_j}^j(\lambda_0) = 0$. Then every Π_l , $l = 0, \dots, m-1$, has λ_0 as its zero. Indeed, if $i \leq l$, then since $\gamma_{n_i+1}^i(\lambda_0) = 0$, we have $\Pi_l(\lambda_0) = 0$. If $i > l$, then $j > i \geq l+1$ and since $\alpha_{n_j}^j(\lambda_0) = 0$, we have $\Pi_l(\lambda_0) = 0$.

Sufficiency:

Let the conditions (i) and (ii) be valid. Then as in the proof of Theorem 4.1, we can consider the matrix $\lambda \begin{bmatrix} I & 0 \end{bmatrix} - M$ in the equivalent form (4.5). Define

$$Y_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{F}^{1 \times p_i}, \quad i = 1, \dots, m-1.$$

Then (4.5) becomes

$$\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}. \tag{5.7}$$

In the necessity part of the proof, we have proved that the condition (ii) is equivalent to the fact that the matrix N has all invariant factors equal to 1. Thus, the matrix (5.7) has all invariant factors equal to 1, as wanted. \square

In the following theorem, we consider the series connection of the linear systems S_i , $i = 1, \dots, m$, in the case when $\text{rank } B_i = n_i$, $i = 2, \dots, m$, and $\text{rank } C_i = n_i$, $i = 1, \dots, m-1$.

THEOREM 5.2. *Let \mathbb{F} be a field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \dots, m$, $B_1 \in \mathbb{F}^{n_1 \times m_1}$, $\text{rank } B_1 = s$. Let $l_i \geq 0$, $i = 1, \dots, m-1$. There exist matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$,*

$i = 1, \dots, m - 1$, such that the matrix

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ X_1 & A_2 & 0 & \ddots & 0 & 0 \\ 0 & X_2 & A_3 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X_{m-1} & A_m & 0 \end{array} \right] \quad (5.8)$$

is controllable and such that $\text{rank } X_i \leq l_i$, $i = 1, \dots, m - 1$, if and only if

- (i) (A_1, B_1) is controllable
- (ii) $\min\{s, l_i, n_i\} \geq \max_{j=i+1, \dots, m} \{r_j\}$, $i = 1, \dots, m - 1$.

Here, by r_i we have denoted the number of nontrivial invariant factors of $\lambda I - A_i$, $i = 1, \dots, m$.

Proof.

Necessity:

Since the matrix (5.8) is controllable, we directly obtain the condition (i). Also, considering the submatrices of (5.8) formed by its last $\sum_{i=1}^j n_{m-i+1}$, $j = 1, \dots, m$, rows, we can apply the result from Theorem 1 in [8] and thus obtain that $s \geq \max_{j=2, \dots, m} \{r_j\}$, and $\text{rank } X_i \geq \max_{j=i+1, \dots, m} \{r_j\}$, $i = 1, \dots, m - 1$. Thus, we obtain the condition (ii).

Sufficiency:

Let $c_1 \geq \dots \geq c_s > 0$ be the nonzero controllability indices of the pair (A_1, B_1) . Let $\alpha_1^i | \dots | \alpha_{n_i}^i$ be the invariant factors of $\lambda I - A_i$, r_i of them nontrivial, and let $D_j^i = d(\alpha_{n_i-j+1}^i)$, $j = 1, \dots, r_i$, $i = 2, \dots, m$. Then the matrix (5.8) is feedback equivalent to the following one

$$\left[\begin{array}{ccccc|c} A_c & 0 & 0 & \ddots & 0 & B_c \\ X'_1 & N(A_2) & 0 & \ddots & 0 & 0 \\ 0 & X'_2 & N(A_3) & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X'_{m-1} & N(A_m) & 0 \end{array} \right] \quad (5.9)$$

where (A_c, B_c) is the Brunovsky canonical form of the pair (A_1, B_1) and

$$N(A_i) = C(\alpha_{n_i-r_i+1}^i) \oplus \dots \oplus C(\alpha_{n_i}^i),$$

is the normal form for similarity of the matrix A_i , $i = 2, \dots, m$, see, e.g. [5].

Now, our problem is equivalent to the problem of defining matrices X'_i , $i = 1, \dots, m - 1$, such that the matrix (5.9) is controllable, and such that $\text{rank } X'_i \leq l_i$, $i = 1, \dots, m - 1$.

Let $l'_i := \min\{s, l_1, \dots, l_i, n_i, n_{i+1}\}$, $i = 1, \dots, m - 1$. We shall define inductively matrices X'_i such that $\text{rank } X'_i = l'_i$, $i = 1, \dots, m - 1$. From the condition (ii) we have

$$l_i \geq \max\{r_{i+1}, \dots, r_m\}, \quad i = 1, \dots, m - 1. \quad (5.10)$$

First we define X'_1 . Let $b_j^1 = \sum_{i=j}^{r_2} D_i^2$, $j = 1, \dots, r_2$. Put r_2 units in the matrix X'_1 at the positions

$$(b_j^1, \sum_{i=1}^{j-1} c_i + 1), \quad j = 1, \dots, r_2.$$

Moreover, let $b_{r_2+1}^1, \dots, b_{l'_1}^1$ be any $l'_1 - r_2$ distinct numbers from the set $\{1, \dots, n_2\} \setminus \{b_1^1, \dots, b_{r_2}^1\}$. Then put $l'_1 - r_2$ units at the positions

$$(b_j^1, \sum_{i=1}^{j-1} c_i + 1), \quad j = r_2 + 1, \dots, l'_1,$$

while all other entries in X'_1 we put to be zeros. Obviously $\text{rank } X'_1 = l'_1$.

Inductively, we define matrices X'_j , $j = 2, \dots, m - 1$:

Let $b_k^j = \sum_{i=k}^{r_{j+1}} D_i^{j+1}$, $k = 1, \dots, r_{j+1}$. Moreover, let $b_{r_{j+1}+1}^j, \dots, b_{l'_j}^j$ be any $l'_j - r_{j+1}$ distinct numbers from the set $\{1, \dots, n_j\} \setminus \{b_1^j, \dots, b_{r_{j+1}}^j\}$. Now, put l'_j units in the matrix X'_j in the rows $b_1^j, \dots, b_{l'_j}^j$ such that they belong to any l'_j different columns among the following ones:

$$\{b_i^{j-1} + 1, \quad i = 2, \dots, l'_{j-1}\} \cup \{1\},$$

while all other entries in X'_j , $j = 2, \dots, m - 1$, we put to be zeros.

Such obtained matrix

$$\left[\begin{array}{ccccc|c} A_c & 0 & 0 & \ddots & 0 & B_c \\ X'_1 & N(A_2) & 0 & \ddots & 0 & 0 \\ 0 & X'_2 & N(A_3) & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X'_{m-1} & N(A_m) & 0 \end{array} \right] \quad (5.11)$$

is controllable, and $\text{rank } X'_i = l'_i \leq l_i$, $i = 1, \dots, m - 1$, as wanted. \square

In order to clarify the way of defining the matrices X'_1, \dots, X'_{m-1} in the previous theorem we give the following example:

EXAMPLE 5.3. Let $m = 3$, $n_1 = 4$, $n_2 = 5$ and $n_3 = 4$. Let $l_1 = 4$, $l_2 = 2$ and $s = 2$. Let (A_1, B_1) be a controllable pair of matrices with $2 \geq 2$ as nonzero controllability indices. Then

$$(A_c, B_c) = \left(\left(\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{array} \right] \right) \right).$$

Let $r_2 = 1$, and let α be the only nontrivial invariant polynomial of A_2 , $d(\alpha) = 5$, i.e. $\alpha = \alpha_5^2$, while $\alpha_1^2 = \dots = \alpha_4^2 = 1$. Thus,

$$N(A_2) = C(\alpha).$$

Let $r_3 = 2$, and let $\beta|\gamma$ be the nontrivial invariant polynomials of A_3 , $d(\beta) = 1$ and $d(\gamma) = 3$, i.e. $\alpha_3^3 = \beta$ and $\alpha_4^3 = \gamma$, while $\alpha_1^3 = \alpha_2^3 = 1$. Thus

$$N(A_3) = C(\beta) \oplus C(\gamma).$$

Then both conditions (i) and (ii) from Theorem 5.2 are satisfied. Now define matrices X'_1 and X'_2 as explained in the theorem:

Since $l'_1 = 2$, $c_1 = 2$ and $b_1^1 = D_1^2 = d(\alpha) = 5$, put a unit in the matrix X'_1 at the position $(5, 1)$. Let $b_2^1 = 4$ ($4 \in \{1, \dots, 5\} \setminus \{5\}$). Then put a unit at the position $(4, 3)$, and all other entries in the matrix X'_1 put to be zeros. Thus,

$$X'_1 = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right].$$

Moreover, since $l'_2 = 2$ and $b_1^2 = D_1^3 + D_2^3 = d(\gamma) + d(\beta) = 4$, $b_2^2 = D_2^3 = d(\beta) = 1$, put units in the matrix X'_2 on the positions $(4, 1)$ and $(1, b_2^1 + 1) = (1, 5)$, and all other entries in the matrix X'_2 put to be zeros. Hence,

$$X'_2 = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right].$$

For such defined X'_1 and X'_2 the matrix

$$\left[\begin{array}{ccc|c} A_c & 0 & 0 & B_c \\ X'_1 & C(\alpha) & 0 & 0 \\ 0 & X'_2 & C(\beta) \oplus C(\gamma) & 0 \end{array} \right] \quad (5.12)$$

is controllable and $\text{rank } X'_1 = 2 \leq 4$, $\text{rank } X'_2 = 2 \leq 2$, as wanted.

As a direct consequence of the previous result we obtain the following theorem:

THEOREM 5.4. *Let \mathbb{F} be a field. Let $A_i \in \mathbb{F}^{n_i \times n_i}$, $i = 1, \dots, m$, and $B_1 \in \mathbb{F}^{n_1 \times m_1}$ be such that the pair (A_1, B_1) is controllable, $\text{rank } B_1 = s$. There exist matrices $X_i \in \mathbb{F}^{n_{i+1} \times n_i}$, $i = 1, \dots, m - 1$, such that*

$$\left[\begin{array}{ccccc|c} A_1 & 0 & 0 & \ddots & 0 & B_1 \\ X_1 & A_2 & 0 & \ddots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & X_{m-1} & A_m & 0 \end{array} \right] \quad (5.13)$$

is controllable if and only if the following conditions are valid:

- (i) $s \geq \max\{r_2, r_3, \dots, r_m\}$
- (ii) $n_i \geq \max\{r_i, \dots, r_m\}, \quad i = 2, \dots, m.$

Here r_i is the number of the nontrivial invariant factors of $\lambda I - A_i$, $i = 2, \dots, m$.

Acknowledgments. Many thanks are given to the referee whose valuable comments have improved the presentation of the paper. The author is supported by *Fundação para a Ciência e a Tecnologia* / (FCT), grant no. SFRH/BPD/26607/2006. When this work started, the author was supported by grant CELC/1431/BPD/06.

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